MECHANICS

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Even Semester

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Unit I **Impact of Elastic Bodies and Motion of Projectile**

Impulse and Impact: Impulse of a force–collision–elastic and inelastic collision– laws of impact–direct impact of two smooth spheres– loss of kinetic energy due ot direct impact–oblique impact of two smooth spheres–loss of kinetic energy due to oblique impact

Projectile motion Theory of projectile motion–range, maximum height, time of flight of the projectile particle–motion of projectile particle down an inclined plane–two body problem–reduced mass

Unit IV(a) **Newtonian Mechanics**

Center of mass–definition–center of mass of two particle system–conservation of linear and angular momentum of a particle–basic idea of degree of freedom–generalized co-ordinates and generalized momentum.

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Impact of Elastic Bodies and Motion of Projectile

Impact of Elastic Bodies

The colliding of two bodies aganinstg each other, or impinging of one body on another , is known as impact. Although the duration of impact is very small, it results in a change in the magnitude and even direction of the velocities of the colliding bodies.

1.1 Impuse of a force

Consider a constant force **F** which acts for a time t on a body of mass m, thus changing its velocity from **u** to **v**. Because the force is constant, the body will travel with constant acceleration **a** where

$$\mathbf{F} = m\mathbf{a}$$

and
$$\mathbf{a}t = \mathbf{v} - \mathbf{u}$$

hence,
$$\frac{\mathbf{F}}{m}t = \mathbf{v} - \mathbf{u}$$

or
$$\mathbf{F}t = m\mathbf{v} - m\mathbf{u}$$

The product of constant force**F** and time t for which it acts is called **impulse(J)** of the force and this is equal to the change in linear momentum it produces.

Thus,
$$\mathbf{J} = \mathbf{F}t = \Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i \tag{1.1.1}$$

Impulsive forces: Definition. An impulsive force is an infinitely great force acting for a very short interval of time, such that their product is finite.Examples for some approximate impulsive forces are

1. The blow of hammer on a pile

2. The force exerted by the bat on the cricket ball

Note: The force and the time cannot be measured because one is too great and the other is too small. Nevertheless, their product, which is finite, is capable of measurement.

1.2 Collision

In collision relatively large force acts on each colliding particle for a relatively short time. The force is called impulsive force. The concept of collision has given much information regarding atoms, nucleus and elementary particles. During the collision, the colliding object may be undergoing physical and non physical contact.

Example of for physical contact collision is billiard ball's collision.Example of non contact collision is scattering of α-particle by atomic nucleus. During the collision relatively strong force acts on the colliding particles and this force has created appreciable effect on the motion of the colliding particles after the collision.

1.2.1 Elastic and Inelastic collision

There are two types of collision,

1. Elastic and

2. Inelastic

Elastic Collison

Elastic collisions are those in which the total kinetic energy before and after collision remains unchanged. Collisions between atomic, nuclear and fundamental particles are the true elastic collisions. Collision between ivory or glass balls can be treated as approximately elastic collisions. In such a collision between particles, we have

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

and

$$\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

where m_1 and m_2 are the respective masses of the two particles and u_1, u_2 and v_1, v_2 are theirs velocity before and after collisions.

Inelastic Collison

If the kinetic energy is not conserved, the collision is said to be inelastic. When two bodies stick togetrher after collision, the collision is said to be completely inelastic.

1.2.2 Law's of impact

Newton's Experimental law of impact–coefficient of restitution

The ratio of relative velocity before and after collision is constant and is in opposite direction. This constant is called **coefficient of restitution**

$$\frac{v_1 - v_2}{u_1 - u_2} = -e$$

where, $(u_1 - u_2)$ and $(v_1 - v_2)$ are their relative velocities, before and after impact. e lies between 0 and 1. if e=0, the bodies are called perfectly plastic bodies. if e=1, the bodies are called perfectly elastic bodies.

Motion of twao smooth bodies perpendicular to the line of impact

There is no change in velocity of a body in a direction perpendicular to the common normal to the impact.

Principle of conservation of momentum

The total momentum of two bodies after impact along the common normal should be equal to the total momentum before the impact along the same direction.

1.2.3 Direct and Oblique impact

Direct Impact: Two bodies are said to be impige directly when the direction of motion of each is along he common normal at the point where they touch.
Oblique Impact: Two bodies are said to impinge obliquely, if the direction of motion of either or both is along the common normal at the point of contact.
Note: The common normal at the point of contact is called line of impact. Thus in the case of two spheres the line of impact is the line joining their centres.

1.3 Direct impact of two smooth spheres:

A smooth sphere of mass m_1 moving with a velocity u_1 impinges on another smooth sphere of mass m_2 moving in the same direction with velocity u_2. If e is the coefficient of restitution between them, fine the velocities of the sphere after impact.

Since the spheres are smooth, there is no impulsive force on either along the common tangent.

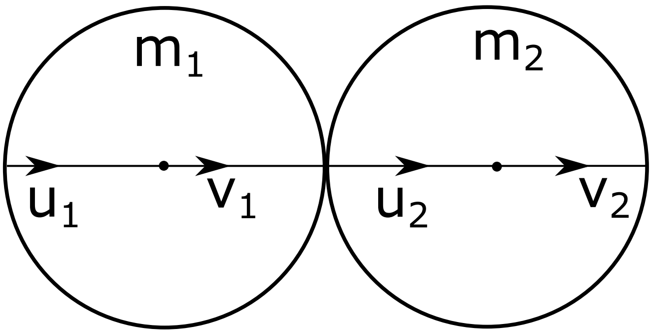

Figure 1.1: Direct collision of two smooth sphere

Hence in this direction their velocities after impact are the same as their original velocities. Let v_1 and v_2 be the velocities of the two spheres along the common normal after impact.

By the principle of conservation of momentum,

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \tag{1.3.1}$$

By Newton's experimental law,

$$v_1 - v_2 = -e(u_1 - u_2) \tag{1.3.2}$$

multiplying equation (1.3.2) by m_2 and adding to (1.3.1)

$$\begin{aligned} v_1(m_1 + m_2) &= m_2u_2(1 + e) + u_1(m_1 - em_2) \\ v_1 &= \frac{m_2u_2(1 + e) + u_1(m_1 - em_2)}{m_1 + m_2} \end{aligned} \tag{1.3.3}$$

multiplying equation (1.3.2) by m_1 and subtracting from equation (1.3.1)

$$\begin{aligned} v_2(m_1+m_2) &= m_1u_1(1+e)+u_2(m_2-em_1) \\ v_2 &= \frac{m_1u_1(1+e)+u_2(m_2+em_1)}{m_1+m_2} \end{aligned} \tag{1.3.4}$$

equation (1.3.3) and (1.3.4) give the velocities of the two sphere after impact.

Cor.1. The impulse of blow on the sphere of mass m_1 =change in momentum produced in it.

$$m_1(v_1-u_1) = \frac{m_1m_2(1+e)(u_2-u_1)}{m_1+m_2}$$

Cor.2. If e=1 and $m_1 = m_2$ then $v_1 = u_2$ and $v_2 = u_1$. Thus, if two equal perfectly, elastic spheres impinge directly, they interchange their velocities.

1.3.1 Loss of K.E due to direct impact of two smooth spheres

Let m_1, m_2 be the masses, u_1 and u_2, v_1 and v_2 their velocities before and after impact and e the coefficient of restitution. Then, by the principle of conservation of linear momentum,

$$m_1v_1+m_2v_2 = m_1u_1+m_2u_2 \tag{1.3.1}$$

By Newton's experimental law,

$$v_1-v_2 = -e(u_1-u_2) \tag{1.3.2}$$

Square equation (1.3.1),

$$\begin{aligned} (m_1v_1+m_2v_2)^2 &= (m_1u_1+m_2u_2)^2 \\ m_1^2v_1^2+m_2^2v_2^2+2m_1m_2v_1v_2 &= (m_1u_1+m_2u_2)^2 \end{aligned} \tag{1.3.3}$$

Square equation (1.3.2) and multiply both sides by m_1 and m_2

$$\begin{aligned} m_1m_2(v_1-v_2)^2 &= e^2m_1m_2(u_1-u_2)^2 \\ m_1m_2v_1^2+m_1m_2v_2^2-2m_1m_2v_1v_2 &= e^2m_1m_2(u_1-u_2)^2 \end{aligned} \tag{1.3.4}$$

adding equations (1.3.3) and (1.3.4), we get

$$m_1^2v_1^2+m_2^2v_2^2+\cancel{2m_1m_2v_1v_2}+m_1m_2v_1^2+m_1m_2v_2^2-\cancel{2m_1m_2v_1v_2} = (m_1u_1+m_2u_2)^2+e^2m_1m_2(u_1-u_2)^2$$

Taking common term on LHS and add and subtract $m_1m_2(u_1-u_2)^2$ on RHS

$$m_1v_1^2(m_1+m_2)+m_2v_2^2(m_1+m_2) = (m_1u_1+m_2u_2)^2+m_1m_2(u_1-u_2)^2+e^2m_1m_2(u_1-u_2)^2-m_1m_2(u_1-u_2)^2$$

$$= m_1^2u_1^2+m_2^2u_2^2+\cancel{2m_1m_2u_1u_2}+m_1m_2u_1^2+m_1m_2u_2^2-\cancel{2m_1m_2u_1u_2}+e^2m_1m_2(u_1-u_2)^2-m_1m_2(u_1-u_2)^2$$

$$m_1v_1^2(m_1+m_2)+m_2v_2^2(m_1+m_2) = (m_1+m_2)m_1^2u_1^2+(m_1+m_2)m_2^2u_2^2+e^2m_1m_2(u_1-u_2)^2-m_1m_2(u_1-u_2)^2 \tag{1.3.5}$$

divide the entire equation with $m_1 + m_2$ and multiply with $\frac{1}{2}$

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 + \frac{1}{2}(e^2 - 1)(u_1 - u_2)^2\frac{m_1m_2}{(m_1 + m_2)}$$

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 - \frac{1}{2}(1 - e^2)(u_1 - u_2)^2\frac{m_1m_2}{(m_1 + m_2)} \tag{1.3.6}$$

Now, $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 =$ K.E after impact

and, $\frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 =$ K.E before impact

Therefore the loss of Kinetic energy $\boxed{= \frac{1}{2} \quad \frac{m_1m_2(u_1 - u_2)^2}{m_1 + m_2}(1 - e^2)}$ (1.3.7)

Note: When e=1, the loss of K.E is zero. In general e¡1 so that $(1 - e^2)$ is positive. Hence, there is always a loss of K.E due to impact. The K.E loss during impact is converted into (i) sound, (ii)heat or (iii) vibration or rotation of the colliding bodies.

when e=0, the loss in K.E $= \frac{1}{2} \quad \frac{m_1m_2(u_1 - u_2)^2}{m_1 + m_2}$

1.4 Oblique impact of two smooth sphere

A smooth sphere of mass m_1 moving with velocity u_1 impinges obliquely on a smooth sphere of mass m_2 moving with velocity u_2. If the direction of motion before impact make angles $\alpha and \beta$ with the common normal, find the velocities and direction of the spheres after impact

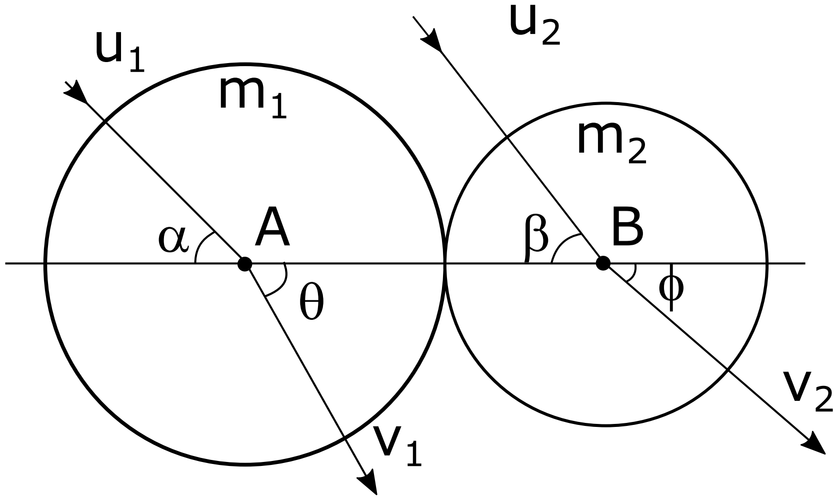

Figure 1.2: Oblique collision of two smooth sphere

Let AB be the common normal (1.2). Let v_1 and v_2 be the velocities of the two spheres after impact making an angle θ and ϕ with common normal AB. Before impact velocities along the common normal AB are $u_1 \cos\alpha$ and $u_2 \cos\beta$ and velocities perpendicular to AB are $u_1 \sin\alpha$ and $u_2 \sin\beta$. After impact velocities along AB are $v_1 \cos\theta$ and $v_2 \cos\phi$ and perpendicular to AB are $v_1 \sin\theta$ and $v_2 \sin\phi$ By the prinicple of conservation of momentum, the total momentum of two spheres along the common normal is unaltered by the impact.

$$\therefore \quad m_1 v_1 \cos\theta + m_2 v_2 \cos\phi = m_1 u_1 \cos\alpha + m_2 u_2 \cos\beta \tag{1.4.1}$$

by Newton's experimental law on relative velocity along the common normal

$$v_1 \cos\theta - v_2 \cos\phi = -e(u_1 \cos\alpha - u_2 \cos\beta) \tag{1.4.2}$$

since there is no force perpendicular to the common normal AB, the velocities of the spheres perpendicular to the common normal AB remain unaltered due ot impact. Hence

$$v_1 \sin\theta = u_1 \sin\alpha \tag{1.4.3}$$

and,

$$v_2 \sin\phi = u_2 sin\beta \tag{1.4.4}$$

multiplying equation (1.4.2) by m_2 and adding to equation (1.4.1)

$$\begin{aligned} m_1 v_1 \cos\theta + \cancel{m_2 v_2 \cos\phi} &= m_1 u_1 \cos\alpha + m_2 u_2 \cos\beta \\ m_2 v_1 \cos\theta - \cancel{m_2 v_2 \cos\phi} &= -e m_2 u_1 \cos\alpha + e u_2 m_2 \cos\beta \\ v_1 \cos\theta &= \frac{u_1 \cos\alpha (m_1 - e m_2) + m_2 u_2 \cos\beta (1+e)}{m_1 + m_2} \end{aligned} \tag{1.4.5}$$

multiplying equation (1.4.2) by m_1 and subtracting from equation (1.4.1)

$$\begin{aligned} \cancel{m_1 v_1 \cos\theta} + m_2 v_2 \cos\phi &= m_1 u_1 \cos\alpha + m_2 v_2 \cos\beta \\ -\cancel{m_1 v_1 \cos\theta} + m_1 v_2 \cos\phi &= e m_1 u_1 \cos\alpha - e u_2 m_1 \cos\beta \\ v_2 \cos\phi &= \frac{m_1 (1+e) v_1 \cos\alpha + (m_2 - e m_1) u_2 \cos\beta}{(m_1 + m_2)} \end{aligned} \tag{1.4.6}$$

Squaring equation (1.4.3) and equation (1.4.5) and adding, we get v_1^2 and hence we can find v_1. Dividing equation (1.4.3) and (1.4.5) ,we get $\tan\theta$. Similarly, from (1.4.4) and (1.4.6) we get v_2 and $\tan\phi$. Therefore v_1, v_2, ϕ and θ are determined uniquely.

Cor.1 impulse is equal to change in momentum measured along its common normal

$$\begin{aligned} &= m_1 v_1 \cos\theta - m_1 u_1 \cos\alpha \\ &= m_1 \left(\frac{(m_1 - e m_2) u_1 \cos\alpha + m_2 (1+e) u_2 \cos\beta}{(m_1 + m_2)} - u_1 \cos\alpha \right) \\ &= \frac{m_1}{(m_1 + m_2)} \left(\cancel{m_i u_1 \cos\alpha} - e m_2 u_1 \cos\alpha + m_2 (1+e) u_2 \cos\beta - \cancel{m_1 u_1 \cos\alpha} - m_2 u_1 \cos\alpha \right) \\ &= \frac{m_1 m_2 (1+e)}{m_1 + m_2} (u_2 \cos\beta - u_1 \cos\alpha) \end{aligned} \tag{1.4.7}$$

This is equal and opposite to the impulse on m_2.

1.4.1 Loss of K.E. due to Oblique impact

The velocities of the spheres perpendicular to the common normal are unaltered. Therefore, the loss of K.E. is the same as in the case of direct impact if we substitute $u_1 \cos\alpha$ and $u_2 \cos\beta$ for u_1 and u_2 respectively.

$$\therefore \quad \text{The loss in K.E.} = \frac{m_1 m_2 (1-e^2)}{2(m_1+m_2)} (u_1 \cos\alpha - u_2 \cos\beta)^2 \tag{1.4.1}$$

1.5 Projectile Motion

Motion of a particle under constant acceleration (acceleration due to gravity) is called Projectile motion. In projectile motion, the particle is either in straight line (One dimensional) or parabolic(Two dimensional). In one dimensional motion, the initial velocity make an angle either zero or 180 with constant acceleration. In parabolic motion the angle is other than zero or 180.

1.5.1 Time of Flight, Maximum Height and Horizontal Range of a Projectile

Figure (1.3) shows a particle projected form the point O with an initial velocity u at an angle α with the horizontal. It goes through the highest point A and falls at B on the horizontal surface through O. The point O is called the **point of projection**, the angle α is called **angle of projection**, the distance OB is called **Horizontal range (R)** or simply range and the vertical height AC is called **Maximum height (H)**. The total time taken by the particle in describing the path OAB is called the **Time of flight(T)**.

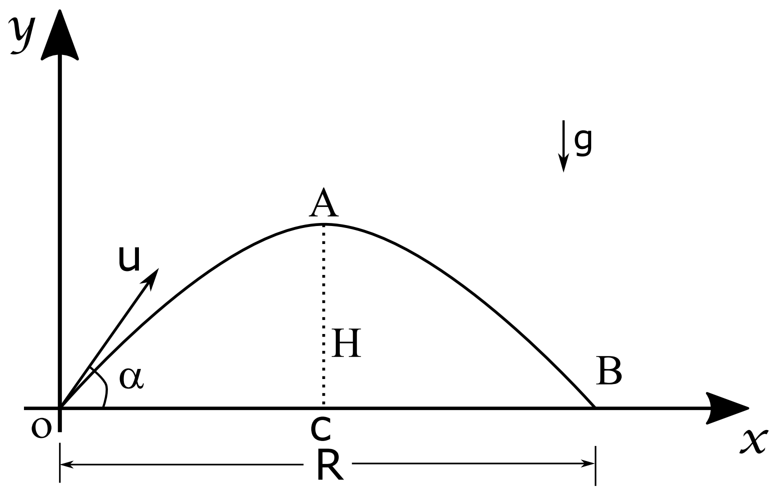

Figure 1.3: Projectile Motion

Time of Flight (T)

Reference to the figure, x and y axis are in the direction shown in Figure. X is along the horizontal direction and y is vertically upwards. Thus,

$$u_x = u\cos\alpha$$
$$u_y = u\sin\alpha$$
$$a_x = 0$$

and
$$a_y = -g$$

At point B, $s_y = 0$. So applying in the equation

$$s_y = u_y t + \frac{1}{2}a_y t^2$$

we have,

$$0 = (u\sin\alpha)t - \frac{1}{2}gt^2$$

$\therefore$
$$t = 0, \quad \frac{2u\sin\alpha}{g}$$

Both $t = 0$ and $t = \frac{2u\sin\alpha}{g}$ corresponding to the situation where $s_y = 0$. and time $t = \frac{2u\sin\alpha}{g}$ corresponding to point B. Thus, time of flight of the projectile is

$$T = t_{OAB} \qquad \text{or} \qquad \boxed{T = \frac{2u\sin\alpha}{g}}$$

Maximum Height(H)

At point A, vertical component of velocity becomes zero, i.e. $v_y = 0$ Substituting the proper values in

$$v_y^2 = u_y^2 + 2a_y s_y$$

we have,
$$0 = (u\sin\alpha)^2 + 2(-g)H$$

$\therefore$
$$\boxed{H = \frac{u^2\sin^2\alpha}{2g}}$$

Horizontal Range (R) Distance OB is the rangae R. This is also equal to the displacement of particle along x-axis in time t=T. Thus, applying $s_x = u_x t + \frac{1}{2}a_x t^2$, we get

$$R = (u\cos\alpha)(\frac{2u\sin\alpha}{g}) + 0$$

as
$$a_x = 0 \text{ and } \qquad t = T = \frac{2u\sin\alpha}{g}$$

$\therefore$
$$R = \frac{2u^2\sin\alpha\cos\alpha}{g} = \frac{u^2\sin 2\alpha}{g}$$

$$\boxed{R = \frac{u^2\sin 2\alpha}{g}}$$

1.5.2 Time period and range on an inclined plane

A particle is projected with velocity u at an angle α to the horizontal form a point O on an inclined plane, inclined at an angle θ to the horizontal. Let the particle strike the inclined plane at A. Then OA in the range on the inclined plane.

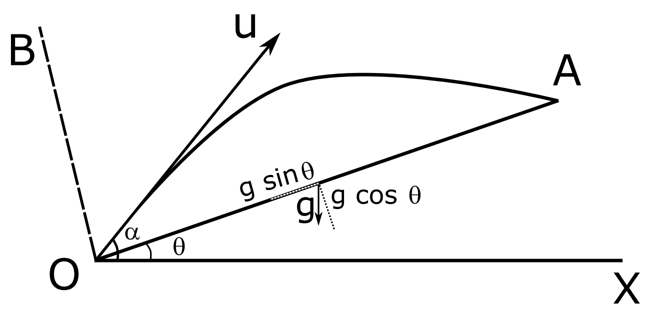

Figure 1.4: Range on inclined plane

Let OX be the horizontal and OA be inclined plane. OB is the perpendicular to OA.

$$\text{Component of initial velocity } u \text{ along OA} = u\cos(\alpha-\theta)$$
$$\text{Component of initial velocity } u \text{ along OB} = u\sin(\alpha-\theta)$$

The projectile move in opposite direction of g

$$\text{Acceleration along OA} = -g\sin\theta$$
$$\text{Acceleration along OB} = -g\cos\beta$$

Now, let T be the time taken by the particle to go from O to A. When the particle reaches A after time T, The distance moved perpendicular to the plane is zero. Hence on substituting equation $s = ut + \frac{1}{2}at^2$, we have

$$0 = u\sin(\alpha-\theta).T - \frac{1}{2}g\cos\theta.T^2$$

$$\therefore \qquad \boxed{T = \frac{2u\sin(\alpha-\theta)}{g\cos\theta}} \tag{1.5.1}$$

When the particle strikes A after time T, the distance OA moved is the range on the inclined plane

$$\therefore R = u\cos(\alpha-\theta).T - \frac{1}{2}g\sin\theta.T^2$$
$$= u\sin(\alpha-\theta)\frac{2u\sin(\alpha-\theta)}{g\cos\theta} - \frac{1}{2}g\sin\theta\frac{4u^2\sin^2(\alpha-\theta)}{g^2\cos^2\theta}$$
$$= \frac{2u^2\sin(\alpha-\theta)}{g\cos^2\theta}\left[\cos(\alpha-\theta)\cos\theta - \sin(\alpha-\theta)\sin\theta\right]$$

$$\boxed{R = \frac{2u^2\sin(\alpha-\theta)\cos\alpha}{g\cos^2\theta}} \tag{1.5.2}$$

1.5.3 Range and Time of flight down an inclined plane

The particle is projected down the inclined plane from O at an elevation α as on Figure(1.5). Initial velocities along and perpendicular to OA are $u\cos(\alpha+\theta)$ and $u\sin(\alpha+\theta)$. Acceleration along and

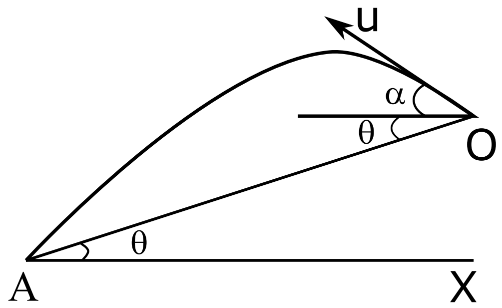

Figure 1.5: Range and Time of flight down an inclined plane

perpendicular to OA are $g\sin\theta$ and $-\,g\cos\theta$. When the particle reaches A after time T, the distance moved perpendicular to the inclined plane is zero. Therefore

$$0 = u\sin(\alpha+\theta).T - \frac{1}{2}g\cos\theta.T^2$$

$$\boxed{T = \frac{2u\sin(\alpha+\theta)}{g\cos\theta}} \tag{1.5.1}$$

$$\begin{aligned} H &= u\cos(\alpha+\theta).T + \frac{1}{2}g\sin\theta.T^2 \\ &= u\,\cos(\alpha+\theta)\,\frac{2u\,\sin(\alpha+\theta)}{g\,\cos\theta} + \frac{1}{2}g\sin\theta\frac{4u^2\sin^2(\alpha+\theta)}{g^2\cos^2\theta} \\ &= \frac{2\,u^2\sin(\alpha+\theta)}{g\,\cos^2\theta}\,[\cos(\alpha+\theta)\cos\theta + \sin\theta\sin(\alpha+\theta)] \\ &= \frac{2\,u^2\sin(\alpha+\theta)}{g\,\cos^2\theta}\,\cos\left((\alpha+\theta)-\theta\right) \end{aligned}$$

$$\boxed{H = \frac{2\,u^2\sin(\alpha+\theta)}{g\,\cos^2\theta}\cos\alpha} \tag{1.5.2}$$

1.6 Two Body Problem and the Reduced Mass

Two body problem effectively reduced to one body problem by introduce the concept of *reduced mass.*. Let us consider two particle of masses m_1 and m_2 , whose instantaneous position vectors with respect to origin O in an inertial reference frame are r_1 and r_2 as on figure (1.6).

The vector distance of m_1 from m_2 is $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$

The particles exert gravitational forces of attraction on each other which act along the vector **r** and are central forces. Let F_12 be the force act on mass m_1 by mass m_2. Then equation of motion for m_1 and m_2 with respect to O becomes

$$m_1\frac{\mathrm{d}^2\vec{r}_1}{\mathrm{d}t^2} = \vec{F}_{12} \qquad \text{and} \qquad m_2\frac{\mathrm{d}^2\vec{r}_2}{\mathrm{d}t^2} = \vec{F}_{21}$$

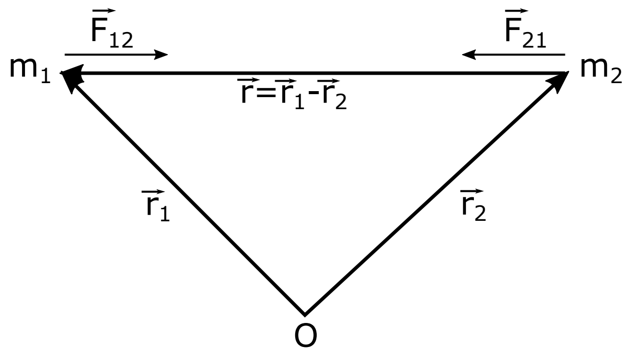

Figure 1.6: Reduced mass in Two body broblem

By Newton's third law, $\vec{F}_{21} = -\vec{F}_{12} = \vec{F}$(say), then

$$\frac{\mathrm{d}^2\vec{r}_1}{\mathrm{d}t^2} = -\frac{\vec{F}}{m_1} \qquad \text{and} \qquad \frac{\mathrm{d}^2\vec{r}_2}{\mathrm{d}t^2} = \frac{\vec{F}}{m_2}$$

Subtracting these equations, we get

$$\frac{\mathrm{d}^2(\vec{r}_1 - \vec{r}_2)}{\mathrm{d}t^2} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\vec{F}$$

But from figure (1.6) $\vec{r}_1 - \vec{r}_2 = \vec{r}$

$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)F\hat{r}$$

Where F is the magnitude of the force and is any function of $\vec{r}$, and $\hat{r}$ is the unit vector along $\vec{r}$

Put
$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

Then,
$$\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -\frac{1}{\mu}F\hat{r}$$

or
$$\mu\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = -F\hat{r}$$

The equation represents a one body problem because it is similar to equation of motion of a single particle of mass μ at a distance $\vec{r}$ from m_1, considered as fixed origin of inertial frame.

Let m_1 and m_2 be the masses of the electron and proton of the hydrogen atom. Their reduced mass is given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1}{1 + (m_1/m_2)} \approx m_1\left(1 - \frac{m_1}{m_2}\right)$$

$\frac{m_1}{m_2}$ is very small in comparison with 1, $\quad \therefore \mu = m_1$.

Chapter 2

Newtonian Mechanics

2.1 Centre of Mass

Definition: Consider the motion of the system consisting of a large number of particles. One point in the system, which behave as whole mass of the system concentrated on it and all external forces acting at this point. This point is called the *Centre of mass* of the system.

2.1.1 Position of Center of Mass of Two Particle

Center of mass of two particles of mass m_1 and m_2 separated by a distance of d lies in between the two particles. The distance of centre of mass from any of the particle (r) is inversely proportional to the mass of the particle (m)

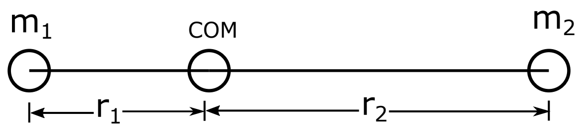

Figure 2.1: Cenre of Mass of two Particles

i.e. $$r \propto \frac{1}{m}$$

or $$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

or $$m_1 r_1 = m_2 r_2$$

or $$\boxed{r_1 = \left(\frac{m_2}{m_1 + m_2}\right) d \quad \text{and} \quad r_2 = \left(\frac{m_1}{m_1 + m_2}\right) d}$$

Here, r_1 is the distance of centre of mass from m_1 and r_2 is the distance of centre of mass form m_1. Further, if $m_1 = m_2$ then r_1 and r_2 is equal to $\frac{d}{2}$. i.e, centre of mass lies midway between the two particles of equal mass. Similarly, $r_1 > r_2$ if $m_1 < m_2$ and $r_1 < r_2$ if $m_1 > m_2$ i.e, centre of mass is nearer to the particle having larger masss.

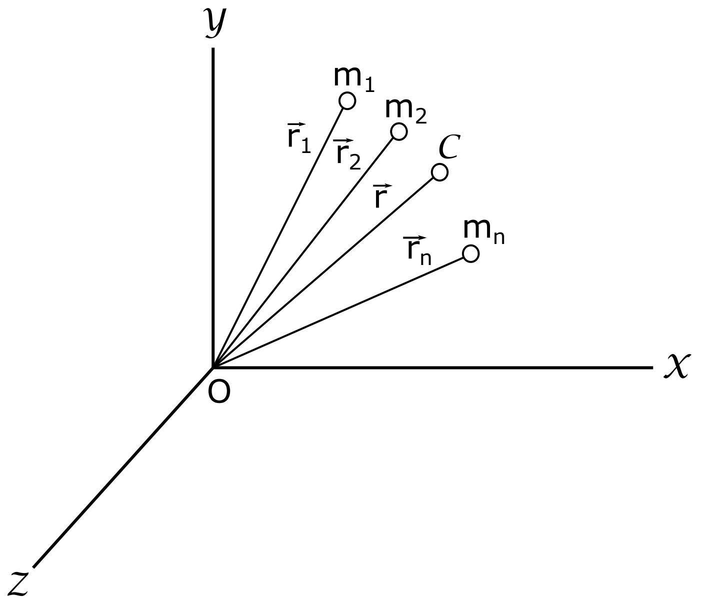

Figure 2.2: Position vector of Center of mass

2.1.2 Position vector of the centre of mass

Let us consider a system of n particles of masses $m_1, m_2, \ldots, m_n$ with position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$ relative to fixed origin Figure (2.2.)

The position vector $\vec{R}$ of the centre of mass of this system is defined by

$$\begin{aligned} \vec{R} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots, m_n\vec{r}_n}{m_1 + m_2 + m_3 + \ldots, m_n} \\ &= \frac{\sum_{k=1}^{n} m_k\vec{r}_k}{\sum_{k=1}^{n} m_k} \\ &= \frac{\sum_{k=1}^{n} m_k\vec{r}_k}{M} \end{aligned}$$

Here, M is the total mass of the system.
Now, $\vec{r}_k = x_k\hat{i} + y_k\hat{j} + z_k\hat{k}$ and $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$
If X,Y and Z be the Cartesian Co-Ordinates of the center of mass, we have

$$\begin{aligned} X &= \frac{m_1x_1 + m_2x_2 + \ldots, m_nx_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{k=1}^{n} m_kx_k}{M} \\ Y &= \frac{m_1y_1 + m_2y_2 + \ldots, m_ny_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{k=1}^{n} m_ky_k}{M} \\ Z &= \frac{m_1z_1 + m_2z_2 + \ldots, m_nz_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{k=1}^{n} m_kz_k}{M} \end{aligned}$$

Here (x_i, y_i, z_i) are co-ordinates of a particle of mass m_1.
For a continuous body, we suppose that the body is formed of a large number of infinitesimal mass elements.Let dm be the mass of such an element at position*(x,y,z)*. Then the co-ordinates of the center of mass are given by

$$X = \tfrac{1}{M}\int_v x \text{ d}m \qquad Y = \tfrac{1}{M}\int_v y \text{ d}m \qquad Z = \tfrac{1}{M}\int_v z \text{ d}m$$

Let $\vec{R}$ be the position vector of the centre of mass of the body. Then

$$\vec{R} = \tfrac{1}{M} \int_v \vec{r} \, \mathrm{d}m$$

2.2 Conservation of Linear momentum

Linear momentum of a particle is defined as the product of its mass and velocity. When a particle of mass m is moving with velocity $\vec{v}$, its linear momentum $\vec{p}$ is given by

$$\vec{p} = m\vec{v}$$

It is a vector quantity. Its units are $kg\ ms^{-1}$ and dimensions are $[MLT^{-1}]$.

If the external force applied to a particle is zero, we have

$$\vec{F} = \tfrac{\mathrm{d}\vec{p}}{\mathrm{d}t} = 0$$

$$\therefore \qquad \vec{p} = m\vec{v} = \text{a constant}$$

i.e, in the absence of an external force, the linear momentum of the particle is remains constant. This is known as the law of conservation of linear momentum.

2.3 Angular Momentum

Consider a particle of mass m and linear momentum $\vec{p}$ at a position $\vec{r}$ relative to origin O. The angular momentum $\vec{L}$ of the particle with respect to the origin O is defined as

$$\vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

Angular momentum is a vector. Its magnitude is given by

$$\vec{L} = rp \sin\theta$$

where, θ is the angle between $\vec{r}$ and $\vec{p}$. Its direction is normal to the plane formed by $\vec{r}$ and $\vec{p}$. The direction is given by the right hand rule.

The unit of angular momentum is $kg\ m^2 s^{-1}$. For circular motion $v = r\omega$. The magnitude of L is $mr^2\omega = I\omega$

2.3.1 Conservation of Angular Momentum

$$\vec{\tau}_{ext} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}$$

suppose there is no external torques acting on a body,$\vec{\tau}_{ext} = 0$ then $\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = 0$ or $\vec{L} = $ a constant. The principle of conservation of angular momentum stated as

When the resultant external torque acting on a system is zero, the total vector angular momentum of the system remains constant.

2.4 Degree of Freedom

The number of mutually independent variables required to define the state or position of a system is the number of degree of freedom. For example, the positions of a simple ideal mass point in space is defined completely by the three cartesian coordinates. it has three degree of freedom. Extending this idea, for a system of N particles moving independently of each other, the number of degree of freedom is 3N.

2.4.1 Constraints

Constraints are restrictions imposed on the position or motion of a system, because of geometrical conditions.
Examples

1. The beads of an abacus are constrained to one dimensional motion by the supporting wires
2. Gas molecules within a container are constrained by the wallls of the vessel to move only inside the container.
3. A particle placed on the surface of a solid sphere is restricted by the constraints so that it can only move on the surface or in the region exterior to the sphere.

2.5 Generalized Co-ordinates

The system consisting of N particles, free from constraints, has 3N independent coordinates or degree of freedom. If the sum of the constrain of all the particles is k, then the system may be regarded as a collection of free particles subjected to (3N-k) independent degree of freedom. So only (3N-k) co-ordinates are needed to describe the motion of the system. These new new co-ordinates $q_1, q_2, \ldots q_k$ are called generalized Co-ordinates of Lagrange. Generalized coordinates may be lengths or angles or any other set of independent quantities which define the position of the system.

Definition: *The generalised coordinates of a material system are the independent parameters* $q_1, q_2, \ldots q_k$ *which completely specify the configuration of the system, i.e., the position of all its particles with respect to the frame of reference*

Generalized co-ordinates are not unique. They may or may not have the dimension of length. Depending on the problem, we chose our convenient co-ordinates with dimensions of energy, Length^2, sometimes the combination of angle and co-ordinates etc.,

2.6 Generalized Momenta

The linear momentum of a particle of mass 'm' moving with velocity $\dot{x}$ is $m\dot{x}$. Its kinetic energy is $T = 1/2m\dot{x}^2$. Differentiating T with respect to $\dot{x}$, we have

$$\frac{\partial T}{\partial \dot{x}} = m\ddot{x} = \vec{p} \tag{2.6.1}$$

We define generalized momentump_i corresponding to generalized co-ordinates q_i as

$$\vec{p_i} = \frac{\partial T}{\partial \dot{x}} \tag{2.6.2}$$

Sometimes it is also known as **conjugate momentum**.

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