Impulse and Impact of Elastic Bodies

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1 Impulse of a Force

The impulse of a force is defined as the product of a force F acting on a body and the interval of time t during which the force acts, that is

$$I = Ft \tag{1}$$

This impulse is often measured by the change in momentum produced in the body because of the force.

Proof: From Newton's second law of motion

$$F(t') = m \times \frac{dv}{dt'} \tag{2}$$

If this force were to act on a body for a time dt', then the impulse

$$dI = F(t')dt' \tag{3}$$

Substituting Eqn. 2 in 3 $\Rightarrow$

$$\begin{aligned} dI &= m\left(\frac{dv}{dt'}\right)dt' \qquad \text{or} \\ I &= \int_0^t m\left(\frac{dv}{dt'}\right)dt' \qquad \text{or} \\ I &= m\,[v]_0^t \qquad \text{or} \\ I &= m\{v(t) - v(0)\} \end{aligned} \tag{4}$$

Let $v(0) = u$ be the initial velocity and $v(t) = v$ be the final velocity. Then Eqn. 4 becomes

$$\begin{aligned} I &= mv - mu, \qquad \text{or} \\ I &= P_f - P_i, \end{aligned} \tag{5}$$

where P_i is the initial momentum and P_f is the final momentum of the body. Thus the impulse of the force equals the change in the momentum of the body. Hence the proof.

2 Impulsive Force

An enormous force acting on a body for a very short duration of time, producing a finite change in the body is called as an *impulsive force*. Examples of this force are

1. The force exerted by a cricketer's bat onto the ball.

2. The blow of a hammer on a nail. etc..

*Note:*The effect produced on a body by an impulsive force is always detected by measuring the change in momentum of the body.

3 Impact

When two bodies strike or hit each other, each body will be exerting a force on the other. From Newton's III law of motion, these two forces will be equal in magnitude but opposite in direction. This process of hitting or striking of two bodies with each other and the resultant change caused in them is called as *Impact*.

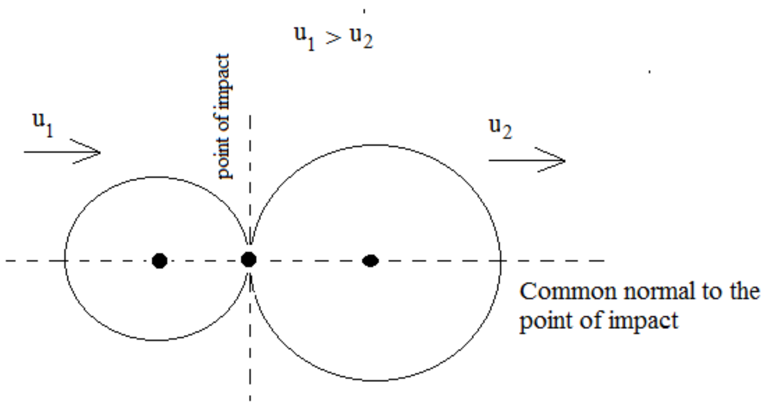

Figure 1: Impact of two smooth spheres

Note:

1. The common meeting point of the two bodies, when they strike each other is called as the *point of impact.*

2. The line joining the centres of the bodies and the point of impact is called as *common normal to the point of impact.*

3.1 Laws of Impact

When two bodies impact on each other, the following laws are observed. These are called as *Laws of Impact.* They are,

I Law: Law of Conservation of Momentum
The total momentum of the two bodies after impact measured along the common normal must be equal to the total momentum before impact measured along the same direction. This is called as the *Law of Conservation of Momentum.*

II Law: Law of Restitution
If u_1 and u_2 be the initial velocities of the two bodies before impact and v_1 and v_2 be the final velocities after impact, then the ratio of their difference is a constant called as the *coefficient of restitution (e).* It is given as,

$$e = -\frac{(v_1 - v_2)}{(u_1 - u_2)} \tag{6}$$

This is called as the *Law of Restitution.*

III Law: Law of Normal Velocities
The change in the velocities of the two bodies perpendicular to the common normal is zero. This is called as *Law of Normal Velocities.*

3.2 Direct and Oblique Impact

Let us consider the impact between two bodies, say two smooth spheres. If the direction of motion of each body before impact is along the common normal to the point of impact, the impact is called as a *direct impact.* If the direction of motion of one or both bodies before impact is inclined to the common normal to the point of impact, then it is called as an *oblique Impact.*

4 Direct Impact Between Two Smooth Spheres

Let us consider two spheres A & B of masses m_1 & m_2 moving with initial velocities u_1 & u_2, directed along the same direction. Let us assume the sphere A is striking the second sphere B, or it is impacting on the second sphere. As the directions of velocities of both the spheres are same, we consider the impact as a *direct impact*. This is shown is Fig. (2).

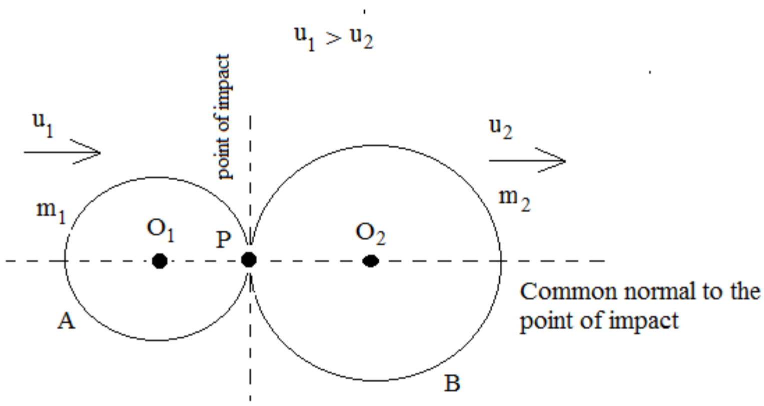

Figure 2: Direct Impact of two smooth spheres A and B

After impact, let us assume the velocities change to v_1 & v_2. Then according to the law of conservation of momentum, the total momentum of the spheres after impact must be equal to the total momentum of the two spheres before impact, that is

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \tag{7}$$

According to the *law of restitution*

$$e = -\frac{(v_1 - v_2)}{(u_1 - u_2)}$$

Rearranging this $\Rightarrow$

$$(v_1 - v_2) = -e(u_1 - u_2) \tag{8}$$

To find the velocity v_1:

This can be done by eliminating or (removing) v_2 from the Eqn. 7. For this we multiply Eqn. (8) by m_2 such that

$$m_2 v_1 - m_2 v_2 = -e m_2 u_1 + e m_2 u_2 \tag{9}$$

Adding Eqn. (7) and Eqn. (9) $\Rightarrow$

$$(m_1 + m_2) v_1 = m_2 u_2 (1 + e) + u_1 (m_1 - e m_2) \qquad \text{or}$$

$$v_1 = \frac{m_2 u_2 (1 + e) + u_1 (m_1 - e m_2)}{(m_1 + m_2)} \tag{10}$$

To find the velocity v_2:

This can be done by eliminating or (removing) v_1 from the Eqn. 7. For this we multiply Eqn. (8) by m_1 such that

$$m_1 v_1 - m_1 v_2 = -e m_1 u_1 + e m_1 u_2 \tag{11}$$

Subtracting Eqn. (11) from Eqn. (7) $\Rightarrow$

$$(m_1 + m_2) v_1 = m_1 u_1 (1 + e) + u_2 (m_2 - e m_1) \qquad \text{or}$$

$$v_2 = \frac{m_1 u_1 (1 + e) + u_2 (m_2 - e m_1)}{(m_1 + m_2)} \tag{12}$$

Equations (10) and (12) give the expressions for the final velocities of the two spheres A and B after a direct impact.

COROLLARY (i)

If the two spheres are of equal mass, that is if $m_1 = m_2$ and if the impact is elastic, that is the coefficient of restitution $e = 1$, then equation (10) and equation (12) reduce to

$$\begin{aligned} v_1 &= u_2 \\ v_2 &= u_1 \end{aligned}$$

This means that *in the case of an elastic collision, the two smooth spheres interchange their velocities after a direct impact.*

COROLLARY (ii)

If the impact is inelastic, then the coefficient of restitution $e = 0$. Hence equations (10) and (12) reduce to

$$\begin{aligned} v_1 &= \frac{m_2u_2 + m_1u_1}{m_1 + m_2} \\ v_2 &= \frac{m_1u_1 + m_2u_2}{m_1 + m_2} \end{aligned}$$

This means that *in the case of an inelastic collision, both the spheres have the same final velocity after a direct impact.*

5 Loss of Kinetic Energy Due to Direct Impact of Two Smooth Spheres

Let the impulse of the impact of the sphere A on the sphere B be the given as

$$I_1 = m_1(v_1 - u_1) \tag{13}$$

This impulse is directed along the common normal in the direction $\overrightarrow{O_1O_2}$. Then the change in kinetic energy of sphere A due to impact is

$$\begin{aligned} \triangle\, T_1 &= \frac{1}{2}m_1v_1^2 - \frac{1}{2}m_1u_1^2 \\ &= \frac{1}{2}m_1(v_1^2 - u_1^2) \\ \triangle\, T_1 &= \frac{1}{2}m_1(v_1 - u_1)(v_1 + u_1). \end{aligned} \tag{14}$$

Substituting Eqn. (13) in Eqn. (14) $\Rightarrow$

$$\triangle\, T_1 = \frac{1}{2}I_1(v_1 + u_1) \tag{15}$$

Similarly let the impulse of the impact of sphere B on sphere A be given as

$$I_2 = m_2(v_2 - u_2) \tag{16}$$

This impulse is directed along $\overleftarrow{O_1O_2}$, that is, opposite to the direction of the impulse I_1. The change in kinetic energy of sphere B can therefore be given similar to Eqn. (15) as

$$\triangle\, T_2 = \frac{1}{2}I_2(v_2 + u_2) \tag{17}$$

The total change in the kinetic energy of the two spheres is

$$\triangle T = \triangle T_1 + \triangle T_2 \tag{18}$$

Substituting Eqn. (15) and(17) in Eqn. (18) $\Rightarrow$

$$\triangle T = \frac{1}{2}[I_1(v_1 + u_1) + I_2(v_2 + u_2)] \tag{19}$$

But $I_2 = -I_1$. Hence

$$\begin{aligned} \triangle T &= \frac{1}{2}I_1[(v_1 + u_1) - (v_2 + u_2)] \\ \triangle T &= \frac{1}{2}I_1[(v_1 - v_2) + (u_1 - u_2)] \end{aligned} \tag{20}$$

We know that the coefficient of restitution is given as

$$e = -\frac{(v_1 - v_2)}{(u_1 - u_2)} \tag{21}$$

Rearranging Eqn. (21) $\Rightarrow$

$$(v_1 - v_2) = -e(u_1 - u_2) \tag{22}$$

Substituting Eqn. (22) in Eqn. (20) $\Rightarrow$

$$\begin{aligned} \triangle T &= \frac{1}{2}I_1[\{-e(u_1 - u_2)\} + (u_1 - u_2)] \\ \triangle T &= \frac{1}{2}I_1(u_1 - u_2)(1 - e) \end{aligned} \tag{23}$$

Equation (23) gives the expression for the loss of kinetic energy due to the direct impact of two spheres in terms of the impulse suffered by the first sphere.

To find an expression for I_1:

We know that the final velocity of the sphere A is given as

$$v_1 = \frac{m_2 u_2(1 + e) + u_1(m_1 - e m_2)}{(m_1 + m_2)} \tag{24}$$

Substituting the above equation in Eqn. (13) $\Rightarrow$

$$\begin{aligned} I_1 &= m_1 \left\{ \frac{m_2 u_2 (1+e) + u_1 (m_1 - e m_2)}{(m_1 + m_2)} \right\} - m_1 u_1 \\ &= \frac{m_1 m_2 u_2 (1+e) + m_1^2 u_1 - m_1 m_2 u_1 e - m_1^2 u_1 - m_1 m_2 u_1}{m_1 + m_2} \\ &= \frac{m_1 m_2 u_2 (1+e) - m_1 m_2 u_1 (1+e)}{m_1 + m_2} \\ I_1 &= \frac{m_1 m_2 (1+e)(u_2 - u_1)}{m_1 + m_2} \qquad (25) \end{aligned}$$

The above equation gives the expression for impulse on the first sphere A due to a direct impact.

Simplified Expression for the Loss of Kinetic Energy $\triangle T$:

Substituting Eqn. (25) in Eqn. (23)$\Rightarrow$

$$\triangle T = \frac{1}{2} \left\{ \frac{m_1 m_2 (1+e)(u_2 - u_1)}{m_1 + m_2} \right\} (u_1 - u_2)(1 - e)$$

or

$$\triangle T = - \left\{ \frac{m_1 m_2 (1 - e^2)(u_1 - u_2)^2}{2(m_1 + m_2)} \right\} \qquad (26)$$

Equation (26) gives the simplified expression for the total change in the kinetic energy due to impact between two smooth spheres. The negative sign indicates that a loss in kinetic energy occurs.

COROLLARY (i)

If the impact is elastic, that is the coefficient of restitution $e = 1$, then

$$\triangle T = 0 \qquad (27)$$

This means that, *in an elastic collision there is no loss in kinetic energy after a direct impact of two smooth spheres or the kinetic energy is conserved in a direct impact of two smooth spheres.*

COROLLARY (ii)

On the other hand, if the impact is inelastic, the coefficient of restitution $e = 0$. Then the loss in kinetic energy is

$$\triangle T = - \left\{ \frac{m_1 m_2 (u_2 - u_1)^2}{2(m_1 + m_2)} \right\} \qquad (28)$$

Thus *in an inelastic collision, the two spheres suffer a finite loss in kinetic energy after a direct impact.*

6 Oblique Impact Between Two Smooth Spheres

Let us consider a smooth sphere, say A of mass m_1 moving with a velocity u_1 impinging obliquely on a sphere B of mass m_2 , moving with a velocity u_2. Let the direction of motion of the spheres, before impact be inclined at angle α and β with the common normal to their point of contact. Let after the oblique impact, their velocities change to v_1 and v_2, with their directions inclined at angles θ and ϕ respectively with the common normal. This is shown in Fig.(3).

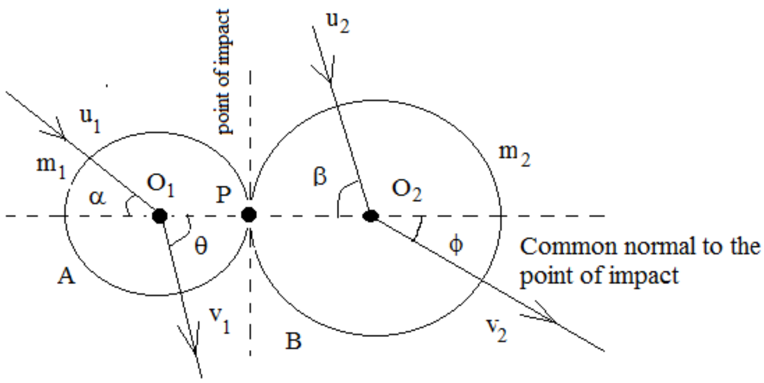

Figure 3: Oblique Impact between two smooth spheres A and B

By the law of conservation of momentum, the total momentum of the spheres along the common normal after an impact must be equal to the total momentum before impact along the same direction, that is

$$m_1 v_{1x} + m_2 v_{2x} = m_1 u_{1x} + m_2 u_{2x} \tag{29}$$

where the suffix x denotes the horizontal component of the velocities. According to the *law of restitution*

$$e = -\frac{(v_{1x} - v_{2x})}{(u_{1x} - u_{2x})}$$

Rearranging this $\Rightarrow$

$$(v_{1x} - v_{2x}) = -e(u_{1x} - u_{2x}) \tag{30}$$

Multiplying Eqn. (8) by $m_2 \Rightarrow$

$$m_2v_{1x} - m_2v_{2x} = -em_2u_{1x} + em_2u_{2x} \tag{31}$$

Adding Eqn. (29) and Eqn. (31) $\Rightarrow$

$$(m_1 + m_2)v_{1x} = m_2u_{2x}(1 + e) + u_{1x}(m_1 - em_2) \tag{32}$$

The horizontal components of the velocities of the two spheres A and B before and after impact are given as

$$\begin{aligned} u_{1x} &= u_1cos\alpha \\ u_{2x} &= u_2cos\beta \\ v_{1x} &= v_1cos\theta \\ v_{2x} &= v_2cos\phi \end{aligned} \tag{33}$$

6.1 To find the velocity v_1 of the sphere A

Substituting Eqns. (33) in Eqn. (32) $\Rightarrow$

$$(m_1 + m_2)v_1cos\theta = m_2u_2cos\beta(1 + e) + u_1cos\alpha(m_1 - em_2) \tag{34}$$

The vertical components of the velocities of the two spheres A and B before and after impact are given as

$$\begin{aligned} u_{1y} &= u_1sin\alpha \\ u_{2y} &= u_2sin\beta \\ v_{1y} &= v_1sin\theta \\ v_{2y} &= v_2sin\phi \end{aligned} \tag{35}$$

Here the subscript y denotes the vertical component of the velocities. According to the *law of the normal velocities*, the change in the velocities in the direction perpendicular to the common normal is zero. This means

$$\begin{aligned} v_{1y} - u_{1y} &= 0 \\ v_{2y} - u_{2y} &= 0. \end{aligned} \tag{36}$$

Hence

$$\begin{aligned} v_{1y} &= u_{1y} \\ v_{2y} &= u_{2y}. \end{aligned} \tag{37}$$

Substituting Eqns. (35) in Eqn. (37) $\Rightarrow$

$$v_1sin\theta = u_1sin\alpha \tag{38}$$

$$v_2sin\phi = u_2sin\beta. \tag{39}$$

Case(i)

For an elastic impact, the coefficient of restitution $e = 1$ and if the two spheres are of the same masses, that is if $m_1 = m_2 = m$, then Eqn(34) becomes

$$2mv_1 cos\theta = 2mu_2 cos\beta$$

or

$$v_1 cos\theta = u_2 cos\beta \tag{40}$$

Squaring and adding Eqns. (38) and (40) $\Rightarrow$

$$v_1^2(cos^2\theta + sin^2\theta) = u_1^2 sin^2\alpha + u_2^2 cos^2\beta$$

or

$$v_1 = \sqrt{u_1^2 sin^2\alpha + u_2^2 cos^2\beta} \tag{41}$$

Case(ii)

For an inelastic impact, the coefficient of restitution $e = 0$. Further if $m_1 = m_2 = m$ then eqn. (34) becomes,

$$2mv_1 cos\theta = m(u_1 cos\alpha + u_2 cos\beta)$$

or

$$v_1 cos\theta = \frac{1}{2}[u_1 cos\alpha + u_2 cos\beta] \tag{42}$$

Squaring and adding Eqns. (42) and (40) $\Rightarrow$

$$v_1^2(cos^2\theta + sin^2\theta) = u_1^2 sin^2\alpha + \frac{1}{4}[u_1 cos\alpha + u_2 cos\beta]^2$$

$$v_1 = \sqrt{u_1^2 sin^2\alpha + \frac{1}{4}[u_1 cos\alpha + u_2 cos\beta]^2} \tag{43}$$

6.2 To find the velocity v_2 of the sphere B

By a similar argument the velocity for the sphere B can be derived.

Case(i)

For an elastic impact, the coefficient of restitution $e = 1$ and if the two spheres are of the same masses, that is if $m_1 = m_2 = m$, then the velocity of the sphere B is given as

$$v_2 = \sqrt{u_1^2 cos^2\alpha + u_2^2 sin^2\beta} \tag{44}$$

Case(ii)

For an inelastic impact, the coefficient of restitution $e = 0$ and if the two spheres are of the same masses, that is if $m_1 = m_2 = m$, then the velocity of the sphere B is given as

$$v_2 = \sqrt{u_2^2 sin^2\beta + \frac{1}{4}[u_1 cos\alpha + u_2 cos\beta]^2} \tag{45}$$

7 Loss of Kinetic Energy Due to Oblique Impact of Two Smooth Spheres

We know that the loss of kinetic energy due the direct impact of two spheres along the common normal is

$$\triangle\, T = -\left\{\frac{m_1 m_2 (1-e^2)(u_1-u_2)^2}{2(m_1+m_2)}\right\} \tag{46}$$

For oblique impact the horizontal components of the velocities should be considered. Hence replacing u_1 by $u_1 cos\alpha$ and u_2 by $u_2 cos\beta$, Eqn. (46) becomes

$$\triangle\, T = -\left\{\frac{m_1 m_2 (1-e^2)(u_1 cos\alpha - u_2 cos\beta)^2}{2(m_1+m_2)}\right\} \tag{47}$$

Note: There is no change in the kinetic energy due to the oblique impact of the two spheres along the direction perpendicular to the common normal.

COROLLARY (i)

For an elastic impact, the coeffecient of restitution $e = 1$. Hence from Eq. 47,

$$\triangle\, T = 0. \tag{48}$$

Thus we find that *in an oblique collision of two smooth spheres, if the collision is elastic, the kinetic energy is conserved.*

COROLLARY (ii)

For an inelastic impact, the coeffecient of restitution $e = 0$. Hence from Eq. 47,

$$\triangle\, T = -\left\{\frac{m_1 m_2 (u_1 cos\alpha - u_2 cos\beta)^2}{2(m_1+m_2)}\right\}. \tag{49}$$

Thus we find that *in an oblique collision of two smooth spheres, if the collision is inelastic, there is a finite loss in kinetic energy.*