Projectile Motion

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1 Introduction

The motion of a body that is projected from a point of the earth so as to move along a predetermined path is called as projectile motion. Examples of projective motion are

1. Firing of rockets.
2. Firing of missiles to hit enemy targets etc..,

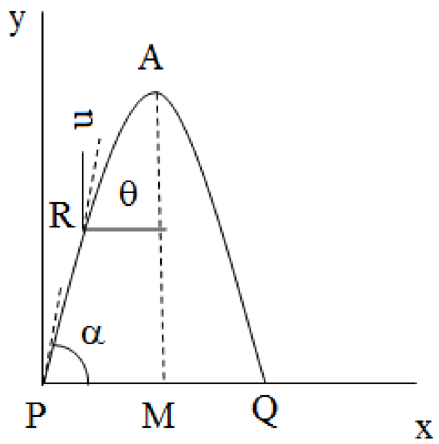

Figure 1: Projectile Motion

To study projectile motion, the following assumptions are made.

1. The resistance offered by air do the projected particle is negligibly small.
2. The acceleration due to gravity g remains a constant throughout the path of motion.

2 Parameters of Projectile Motion

Some of the quantities involved in projectile motion are listed below.

1. **Angle of Projection** (α)**:**
   The angle which the direction of the projection makes with the horizontal axis is called as the *angle of projection* α.

2. **Trajectory of the Projectile Particle:**
   The path described by the particle during its motion is called as *trajectory of the particle.*

3. **Range of the Particle (R):**
   The distance between the point of projection and the point at which the projectile particle meets the horizontal plane is called as the *range of the particle.*

4. **Time of Flight (T):**
   The interval of time from the instant of projection to the instant the particle hits the horizontal plane is called as the *time of flight.*

5. **Greatest Height (h):**
   The point on the trajectory which has the maximum distance from the horizontal plane is called as the *greatest height.*

3 Theory of Projectile Motion

For a particle executing projectile motion, we find that the path described by the particle is a parabola. To study projectile motion mathematically, we use three equations of the kinematics of motion, namely

$$v = u + at \tag{1}$$

$$s = ut + \frac{1}{2}at^2 \tag{2}$$

$$v^2 = u^2 + 2as \tag{3}$$

where

$u \longrightarrow$ initial velocity,

$v \longrightarrow$ final velocity,

$t \longrightarrow$ time of flight,

$a \longrightarrow$ acceleration and

$s \longrightarrow$ displacement

of the projectile particle.

3.1 Trajectory or Path of a Projectile Particle

Let us consider a particle projected from a point on the ground with a velocity u along a direction making an angle α with the horizontal as shown in Fig. (1). The velocity u of the particle can be resolved into horizontal and vertical components as

$$u_h = u\,cos(\alpha) \tag{4}$$
$$u_v = u\,sin(\alpha). \tag{5}$$

The horizontal and vertical distances travelled by the particle can be determined using Eqn. (2). For the distance along the horizontal direction we consider, the displacement $s \equiv x$, and acceleration $a = 0$ while for the vertical direction, the displacement is taken as $s \equiv y$ and acceleration $a = -g$, where g is the acceleration due to gravity. The negative sign indicates that the projectile is moving against the gravitational force of attraction. Thus from Eqn. (2) we have

$$x = u\,cos(\alpha)\,t \tag{6}$$
$$y = u\,sin(\alpha)\,t - \frac{1}{2}gt^2 \tag{7}$$

Rearranging Eqn. (6), the time taken for the projectile particle to reach a point, say R is

$$t = \frac{x}{u\,cos(\alpha)} \tag{8}$$

Substituting Eqn. (8) in Eqn. (7) gives

$$y = u\,sin(\alpha)\frac{x}{u\,cos(\alpha)} - \frac{1}{2}g\left\{\frac{x}{u\,cos(\alpha)}\right\}^2$$

or

$$\boxed{y = tan(\alpha)\,x - \left\{\frac{g}{2u^2\,cos^2(\alpha)}\right\}x^2} \tag{9}$$

The above equation is similar to the equation of a parabola, namely

$$\boxed{y = ax + bx^2} \tag{10}$$

where we have assumed the coefficients as

$$a = tan(\alpha) \qquad \text{and} \tag{11}$$

$$b = -\left\{\frac{g}{2u^2\, cos^2(\alpha)}\right\} \tag{12}$$

3.2 Expressions for the Parameters of the Projectile Motion

The mathematical expressions for some of the parameters of the projectile particle during its motion can be derived as follows.

1. **Instantaneous Velocity of the Projectile Particle $v(t)$:**
   The velocity of the projectile particle at a point, say R on the trajectory after a time t (refer Fig. (1)) can be given from Eqn. (3). Let us assume $s \equiv y$ and $a = -g$, then substituting Eqn. (7)

$$\begin{aligned} v^2 &= u^2 + 2\,(-g)\left\{u\, sin(\alpha)t - \frac{1}{2}gt^2\right\} \\ & \quad or \\ v &= \sqrt{u^2 - 2ug\, sin(\alpha)\, t + g^2t^2}. \end{aligned} \tag{13}$$

2. **Angle of Inclination (θ):**
   The angle which the direction of the projection makes with the horizontal axis is called as the *angle of projection* α. However this *angle of projection* will change as the projectile particle moves along its trajectory. Hence the angle of inclination at a point R on the trajectory at an instant of time t be given as α. We know $tan(\theta) = \left\{\frac{y}{x}\right\}$. Substituting Eqns. (6) & (7) gives

$$tan(\theta) = \left\{\frac{u\, sin(\alpha)\, t - \frac{1}{2}gt^2}{u\, cos(\alpha)\, t}\right\} \qquad or$$

$$\theta = tan^{-1}\left\{\frac{u\, sin(\alpha) - \frac{1}{2}gt}{u\, cos(\alpha)}\right\}. \tag{14}$$

3. **Greatest Height Reached by the Projectile h:**
   The point on the trajectory which has the maximum distance from the horizontal plane (AM) is called as the *greatest height h*. At the

point of greatest height, the final velocity of the particle is zero. Hence assuming $v = 0$, $u = u_v = u\,sin(\alpha)$, $a = -g$ and $y = h$, Eqn. (3) gives

$$\begin{aligned} 0 &= u_v^2 - 2gh \qquad \text{or} \\ h &= \frac{u_v^2}{2g} \qquad \text{or} \\ h &= \frac{u^2\,sin^2(\alpha)}{2g}. \end{aligned} \tag{15}$$

4. **Time of Flight T:**
The interval of time from the instant of projection to the instant the particle hits the horizontal plane is called as *the time of flight T*. This can be calculated by the relation

$$T = t_1 + t_2 \tag{16}$$

where $t_1 \longrightarrow$ is the time taken for the upward motion of the projectile. For the upward motion the displacement along the vertical direction is given by

$$y = usin(\alpha)t_1 - \frac{1}{2}gt_1^2$$

Differentiating this equation gives

$$\frac{dy}{dt} = u\sin(\alpha) - gt_1$$

If height of the projectile particle is a maximum, then its time derivative should be equal to zero. Hence

$$0 = u\ \sin(\alpha) - gt_1$$

Rearranging the above equation gives

$$t_1 = \left\{\frac{u\,sin(\alpha)}{g}\right\}. \tag{17}$$

For the downward motion, we have $t = t_2$, $u = 0$, $a = g$ and $v = u_y$ or $v = u\ \sin(\alpha)$. Hence from Eqn. (1)

$$t_2 = \left\{\frac{u\,sin(\alpha)}{g}\right\} \tag{18}$$

where $t_2 \longrightarrow$ is the time taken for the downward motion of the projectile.

Therefore substituting Eqns. (17) & (18) in Eqn. (16), we have the total time of flight as

$$T = \frac{u\,sin(\alpha)}{g} + \frac{u\,sin(\alpha)}{g}$$

Therefore, the total time of flight of the projectile particle is

$$T = \frac{2u\,sin(\alpha)}{g} \tag{19}$$

5. **Range of the Particle on the Horizontal Plane R:**
The maximum distance covered by the particle in the horizontal direction during the entire time of its flight is called as the *horizontal range of the particle*. It is given as

$$R = \frac{2u^2 \sin(2\alpha)}{g} \tag{20}$$

Proof:
From Eqn. (10), if $x = R$ and $y = 0$, we have

$$\begin{aligned} 0 &= aR + bR^2 && \text{or} \\ bR^2 &= aR && \text{or} \\ R &= \frac{a}{b} && (21) \end{aligned}$$

Substituting Eqns. (9) and (10) in the Eqn. (20), we have

$$\begin{aligned} R &= \frac{a}{b} \\ &= \frac{\tan(\alpha)}{-\left\{\dfrac{g}{2u^2\,cos^2(\alpha)}\right\}} \\ &= \frac{2u^2\tan(\alpha)\cos^2(\alpha)}{g} && \text{or} \\ R &= \frac{2u^2\sin(\alpha)\cos(\alpha)}{g} \end{aligned}$$

(22)

We know $\sin(\alpha)\cos(\alpha) = \sin(2\alpha)$. Substituting this in the above equation gives

$$R = \frac{2u^2 \sin(2\alpha)}{g}$$

Hence the proof.