# Generalized Curvilinear Coordinate System

September 11, 2018

## 1 Introduction

We are familiar with the three dimensional right handed rectangular coordinate system, with $x$, $y$ and $z$ as the coordinate axes. This coordinate system is very much intuitive and has been used to solve many problems in Newtonian and Relativistic Mechanics. However to solve problems involving spherical symmetry such as the motion of the electrons in an atom or problems involving cylindrical geometry such as in the motion of the charged particles in electromagnetic fields etc., this coordinate system is found to be inappropriate. Hence we go in for a generalized curvilinear system wherein the problems become easily solvable using the method of separation of variables. The ability to transform variables and expressions from cartesian coordinate system to other equivalent coordinate systems is therefor absolutely essential for solving a number of problems in Physics.

## 2 Generalized Curvilinear Coordinates

Let us consider a three dimensional space, defined by three single valued functions, say $u_1$, $u_2$ and $u_3$ along the three directions respectively.

**A Point**

Let $P$ be a point in this space. This point can be represented mathematically by the function $P \equiv P(u_1, u_2, u_3)$.

**Coordinate Surfaces**

A coordinate surface is a two dimensional plane along which any two functions defining the position may change, while the third remains a constant. Thus $u_1 = c_1$, $u_2 = c_2$ and $u_3 = c_3$ define coordinate surfaces along the three directions. For the surface $u_1 = c_1$, the function $u_1$ is a constant equal to $c_1$, while the functions $u_2$ and $u_3$ may vary. Similarly for the surface $u_2 = c_2$, the function $u_2$ is a constant equal to $c_2$, while the functions $u_1$ and $u_3$ may vary, while for the surface $u_3 = c_3$, the function $u_3$ is a constant equal to $c_3$, while the functions $u_1$ and $u_2$ may vary. *Note:* In cartesian coordinate system, we have three mutually perpendicular planes $x = constant$, $y = constant$ and $z = constant$.

**Coordinate Lines**

When two coordinate surfaces intersect each other, they form a line pointing along the third direction. This line of intersection is called as the *coordinate line*. For a three dimensional space, we have three coordinate lines, namely $u_1$, $u_2$ and $u_3$ formed by the intersections of the surfaces $(u_2 \& u_3)$,$(u_1 \& u_3)$ and $(u_1 \& u_2)$ respectively.
*Note:* In cartesian coordinate system, we have three coordinate lines $x$, $y$ & $z$ formed by the intersection of the surfaces $(y, z)$, $(x, z)$ and $(x, y)$ respectively.

**Coordinate Axes**

Tangents drawn to the coordinate lines at the coordinate point $P$ are called as coordinate axes. Thus for the point $P$ we have $a_1$, $a_2$ and $a_3$ as coordinate axes, which are tangents to the coordinate lines $u_1$, $u_2$ and $u_3$ respectively as shown in Fig. 1.

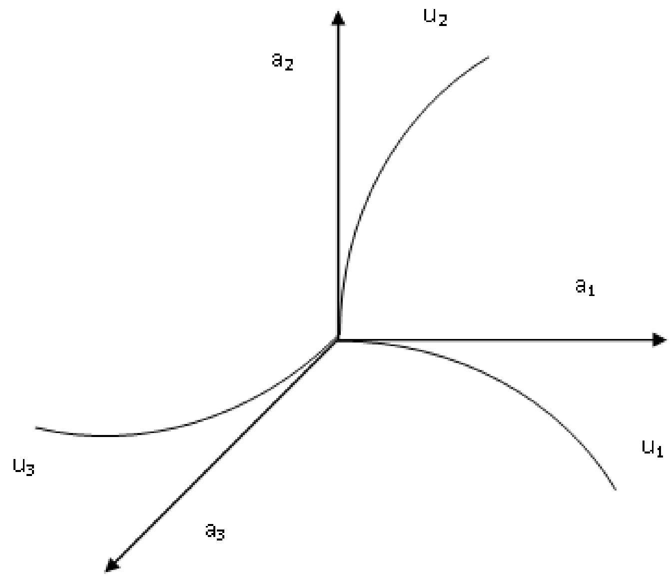


Figure 1: Schematic Representation of Generalized Coordinates

**General Curvilinear Coordinates**

If the relative orientation of the coordinate surfaces change from point to point, then the coordinates $u_1$, $u_2$ and $u_3$ are called as *general curvilinear coordinates.*

**Orthogonal Curvilinear Coordinates**

If the three coordinate surfaces are mutually perpendicular at all points then the coordinates $u_1$, $u_2$ and $u_3$ are called as *orthogonal curvilinear coordinates.*

# 3 Transformation of Curvilinear Coordinates

Curvilinear coordinates obey the following transformation and inverse transformation relations, namely

$$x_i = P_i(u_1, u_2, u_3), i = 1, 2, 3. \tag{1}$$

and

$$u_i = Q_i(x_1, x_2, x_3), i = 1, 2, 3. \tag{2}$$

where $P_i = Q_i^{-1}$ and vice-versa.

**Distance or Displacement**

The position vector of a point may be defined as $\vec{r} \equiv \vec{r}(u_1, u_2, u_3)$. Then an element of displacement of this point may be given as

$$\begin{aligned} d\vec{r} &= \frac{\partial r}{\partial u_1}du_1 + \frac{\partial r}{\partial u_2}du_2 + \frac{\partial r}{\partial u_3}du_3, \\ &= \left\{\frac{\partial r_1}{\partial u_1} + \frac{\partial r_2}{\partial u_1} + \frac{\partial r_3}{\partial u_1}\right\} du_1 + \left\{\frac{\partial r_1}{\partial u_2} + \frac{\partial r_2}{\partial u_2} + \frac{\partial r_3}{\partial u_2}\right\} du_2 + \left\{\frac{\partial r_1}{\partial u_3} + \frac{\partial r_2}{\partial u_3} + \frac{\partial r_3}{\partial u_3}\right\} du_3, \\ &= \sum_{k=1}^{3}\left(\frac{\partial r_k}{\partial u_1}\right) du_1 + \sum_{k=1}^{3}\left(\frac{\partial r_k}{\partial u_2}\right) du_2 + \sum_{k=1}^{3}\left(\frac{\partial r_k}{\partial u_3}\right) du_3, \\ &= \sum_{i=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial r_k}{\partial u_i}\right) du_i \equiv dS. \end{aligned} \tag{3}$$

Here $r_k$ is the $k^{th}$ direction and $\sum_{k=1}^{3}\left(\frac{\partial r_k}{\partial u_i}\right)$ is the incremental change in $r$ along the direction of $u_i$. But $\sum_{k=1}^{3}\left(\frac{\partial r_k}{\partial u_i}\right) = a_i$, where $a_i$ is the coordinate axis along the $i^{th}$ direction. Hence Eq. (3) gives

$$dS = \sum_{i=1}^{3} a_i du_i. \tag{4}$$

**Square of Displacement**

The square of displacement may be given as

$$dS^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i a_j du_i du_j. \tag{5}$$

*Note: For general curvilinear coordinates, $a_i$ and $a_j$ may vary in direction and magnitude from point to point.*

**Metric Coefficient ($g_{ij}$)**

The product of coordinate axes is called as the *metric coefficient* $g_{ij}$. It is given as

$$\begin{aligned} g_{ij} &= a_i.a_j \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\left(\frac{\partial r_k}{\partial u_i}\right)\left(\frac{\partial r_k}{\partial u_j}\right) \end{aligned} \tag{6}$$

In terms of metric coefficients, the square of the displacement becomes

$$dS^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij} du_i du_j. \tag{7}$$

# 4 Orthogonal Curvilinear Coordinates

For orthogonal curvilinear coordinates,

$$g_{ij} = a_i.a_j.\delta_{ij}. \tag{8}$$

This means that if $i \neq j$, $g_{ij} = 0$. Hence for the orthogonal coordinates, the metric coefficients can be given as

$$g_{ij} = \sum_{i=1}^{3}\left(\frac{\partial r_k}{\partial u_i}\right)^2. \tag{9}$$

Then the square of the displacement is given as

$$\begin{aligned} dS^2 &= \sum_{i=1}^{3} g_{ij} du_i^2 \\ &= g_{11}du_1^2 + g_{22}du_2^2 + g_{33}du_3^2. \end{aligned} \tag{10}$$

If $dS_1$, $dS_2$ and $dS_3$ be the length segments of the displacement along the directions of $u_1$, $u_2$ and $u_3$ then the square of the displacement can be given as

$$dS^2 = dS_1^2 + dS_2^2 + dS_3^2. \tag{11}$$

Further the length segments of the displacements along their respective directions are given as

$$\begin{aligned} dS_1 &= h_1 du_1 \\ dS_2 &= h_2 du_2 \\ dS_3 &= h_3 du_3. \end{aligned} \tag{12}$$

where $h_1$, $h_2$ and $h_3$ are called as *scale factors.*

In terms of the components along the three direction, the square of displacements can be given as

$$dS^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2. \tag{13}$$

Comparing Eq. (10) with the above equation Eq. (13) gives

$$\begin{aligned} h_1 &= \sqrt{g_{11}} \\ h_2 &= \sqrt{g_{22}} \\ h_3 &= \sqrt{g_{33}}. \end{aligned} \tag{14}$$

Equations (14) relate the *metric coefficients* and *scale factors.*

***Note*:** Equations (12) define the length elements in curvilinear coordinates. In a similar manner we have the area and volume elements given as

$$d\sigma = h_i h_j du_i du_j. \tag{15}$$

and

$$d\tau = h_i h_j h_k du_i du_j du_k. \tag{16}$$

# 5 Gradient, Divergence, Curl and Laplacian

Let us derive the general expressions for the gradient, divergence, curl and Laplacian operators in the orthogonal curvilinear coordinate system.

## 5.1 Gradient

Let us assume that $\Phi(u_1, u_2, u_3)$ be a single valued scalar function with continuous first order partial derivatives. Then the gradient of $\Phi$ is a vector whose component in any direction $dS_i$, is the derivative of $\Phi$ with respect to $S_i$.

$$\nabla\Phi = \hat{e}_1 \frac{\partial\Phi}{\partial S_1} + \hat{e}_2 \frac{\partial\Phi}{\partial S_2} + \hat{e}_3 \frac{\partial\Phi}{\partial S_3},$$

where $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$ are the unit vectors along $\partial S_1$, $\partial S_2$ and $\partial S_3$ respectively and

$$\frac{\partial\Phi}{\partial S_i} =_{\mathcal{L}t \atop \Delta S_i \to 0} \left\{ \frac{\Phi(S_i + \Delta S_i) - \Phi(S_i)}{\Delta S_i} \right\}$$

But from Eq. (12), $\partial S_i = h_i \partial u_i$. Therefore we have

$$\begin{aligned} \nabla\Phi &= \frac{\hat{e}_1}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial\Phi}{\partial u_3}, \\ &= \left\{ \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3} \right\} \Phi. \end{aligned} \tag{17}$$

From the above equation Eq. (17) we find that the gradient operator itself in orthogonal coordinates is given as

$$\nabla = \left\{ \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3} \right\}. \tag{18}$$

## 5.2 Divergence

Let $\vec{A}$ be a vector in orthogonal curvilinear space. In terms of its components it can be written as

$$\begin{aligned} \vec{A} &= \hat{e}_1 A_1 + \hat{e}_2 A_2 + \hat{e}_3 A_3 \\ &= \sum_{i=1}^{3} \hat{e}_i A_i. \end{aligned} \tag{19}$$

Then the divergence of this vector $\vec{A}$ can be given from Eq. (18) as

$$
\begin{aligned}
\nabla.\vec{A} &= \left\{\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}\right\} . \{\hat{e}_1 A_1 + \hat{e}_2 A_2 + \hat{e}_3 A_3\} \\
&= \sum_{i=1}^{3}\left\{\frac{\hat{e}_i}{h_i}\frac{\partial}{\partial u_i}\right\} . \sum_{j=1}^{3}\{\hat{e}_j A_j\} \\
&= \sum_{i=1}^{3}\frac{1}{h_i}\left(\frac{\partial A_i}{\partial u_i}\right), \qquad \text{since} \quad \hat{e}_i.\hat{e}_j = \delta_{ij} \\
&= \sum_{i=1}^{3}\frac{1}{h_i}\frac{\partial}{\partial u_i}\left\{\frac{1}{h_j h_k}\right\}(h_j h_k A_i) \\
&= \sum_{i=1}^{3}\frac{1}{h_i}\frac{\partial}{\partial u_i}[\Phi\Psi], \qquad \text{where} \quad \Phi = \left\{\frac{1}{h_j h_k}\right\} \qquad \text{and} \quad \Psi = (h_j h_k A_i) \\
&= \sum_{i=1}^{3}\frac{1}{h_i}\left\{\Phi\frac{\partial \Psi}{\partial u_i} + \Psi\frac{\partial \Phi}{\partial u_i}\right\} \\
&= \sum_{i=1}^{3}\frac{1}{h_i}\left\{\Phi\frac{\partial \Psi}{\partial u_i}\right\}, \qquad \text{since} \quad \frac{\partial \Phi}{\partial u_i} = 0, \quad \text{as} \quad \Phi \quad \text{is a scalar} \\
&= \sum_{i,j,k=1}^{3}\frac{1}{h_i}\left\{\frac{1}{h_j h_k}\frac{\partial\,(h_j h_k A_i)}{\partial u_i}\right\}, \\
&= \frac{1}{h_1 h_2 h_3}\frac{\partial\,(h_2 h_3 A_1)}{\partial u_1} + \frac{1}{h_2 h_3 h_1}\frac{\partial\,(h_3 h_1 A_2)}{\partial u_2} + \frac{1}{h_3 h_1 h_2}\frac{\partial\,(h_1 h_2 A_3)}{\partial u_3} \qquad \text{or}
\end{aligned}
$$

$$
\nabla.\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial\,(h_2 h_3 A_1)}{\partial u_1} + \frac{\partial\,(h_3 h_1 A_2)}{\partial u_2} + \frac{\partial\,(h_1 h_2 A_3)}{\partial u_3}\right]. \tag{20}
$$

The above equation (20) gives the expression for the divergence of a vector $\vec{A}$ in a general orthogonal curvilinear coordinate syetem.

## 5.3 Laplacian

We know that the gradient of a scalar function always gives a vector quantity. If $\Phi$ is the scalar function, then the gradient of $\Phi$ is a vector $\vec{A}$ given by

$$\vec{A} = \nabla\Phi. \tag{21}$$

Then comparing Eq. (19) and Eq. (17) we have the components of the vector $\vec{A}$ given by

$$\begin{aligned} A_1 &= \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1} \\ A_2 &= \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2} \\ A_3 &= \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}. \end{aligned} \tag{22}$$

We know

$$\begin{aligned} \nabla^2\Phi &= \nabla.\nabla\Phi \qquad \text{or from Eq.(21)} \\ \nabla^2\Phi &= \nabla.\vec{A} \qquad \text{or from Eq.(20)} \\ \nabla^2\Phi &= \frac{1}{h_1h_2h_3}\left[\frac{\partial\left(h_2h_3A_1\right)}{\partial u_1} + \frac{\partial\left(h_3h_1A_2\right)}{\partial u_2} + \frac{\partial\left(h_1h_2A_3\right)}{\partial u_3}\right]. \end{aligned} \tag{23}$$

Substituting Eq. (22) in Eq. (23), we have

$$\nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right] \tag{24}$$

The Eq. (24) gives the general expression for the Laplacian in orthogonal curvilinear coordinate system.

## 5.4 Curl

Let $\vec{A}$ be a vector in orthogonal coordinate system represented as

$$\vec{A} = \hat{e}_1 A_1 + \hat{e}_2 A_2 + \hat{e}_3 A_3. \tag{25}$$

Multiplying and dividing the components $A_i$ by $h_i$, the above equation (25) becomes

$$\vec{A} = \frac{\hat{e}_1}{h_1}(h_1 A_1) + \frac{\hat{e}_2}{h_2}(h_2 A_2) + \frac{\hat{e}_3}{h_3}(h_3 A_3) \tag{26}$$

Then the curl of $\vec{A}$ can be given as

$$\nabla \times \vec{A} = \nabla \times \left[\frac{\hat{e}_1}{h_1}(h_1 A_1) + \frac{\hat{e}_2}{h_2}(h_2 A_2) + \frac{\hat{e}_3}{h_3}(h_3 A_3)\right]. \tag{27}$$

We know that if $\Phi$ is a scalar and $\Psi$ is a scalar, then

$$\nabla \times (\Phi\Psi) = \Phi\nabla \times \Psi - \Psi \times \nabla\Phi. \tag{28}$$

If $\Phi = (h_1 A_1)$ and $\Psi = \left\{\frac{\hat{e}_1}{h_1}\right\}$, then the fist component of the curl of $\vec{A}$ in Eq. (27) becomes

$$\nabla \times \left[\frac{\hat{e}_1}{h_1}(h_1 A_1)\right] = (h_1 A_1)\nabla \times \left\{\frac{\hat{e}_1}{h_1}\right\} - \left\{\frac{\hat{e}_1}{h_1}\right\} \times \nabla(h_1 A_1). \tag{29}$$

But from vector relations we can prove that

$$(h_1 A_1)\nabla \times \left\{\frac{\hat{e}_1}{h_1}\right\} = 0, \tag{30}$$

Hence Eq. (29) becomes

$$\nabla \times \left[\frac{\hat{e}_1}{h_1}(h_1 A_1)\right] = -\left\{\frac{\hat{e}_1}{h_1}\right\} \times \nabla(h_1 A_1). \tag{31}$$

But from Eq. (18),

$$\nabla = \left\{ \frac{\hat{e}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3} \frac{\partial}{\partial u_3} \right\}. \tag{32}$$

Hence using this Eq. (31) becomes

$$\nabla \times \left[ \frac{\hat{e}_1}{h_1} (h_1 A_1) \right] = - \left\{ \frac{\hat{e}_1}{h_1} \right\} \times \left[ \frac{\hat{e}_1}{h_1} \frac{\partial (h_1 A_1)}{\partial u_1} + \frac{\hat{e}_2}{h_2} \frac{\partial (h_1 A_1)}{\partial u_2} + \frac{\hat{e}_3}{h_3} \frac{\partial (h_1 A_1)}{\partial u_3} \right] \tag{33}$$

But we know that

$$\begin{aligned} \hat{e}_1 \times \hat{e}_1 &= 0 \\ \hat{e}_1 \times \hat{e}_2 &= \hat{e}_3 \\ \hat{e}_1 \times \hat{e}_3 &= -\hat{e}_2. \end{aligned} \tag{34}$$

Hence Eq. (33) becomes

$$\nabla \times \left[ \frac{\hat{e}_1}{h_1} (h_1 A_1) \right] = - \frac{1}{h_1 h_2 h_3} \left[ h_3 \hat{e}_3 \frac{\partial (h_1 A_1)}{\partial u_2} - h_2 \hat{e}_2 \frac{\partial (h_1 A_1)}{\partial u_3} \right]. \tag{35}$$

Similarly evaluating the second and third terms in the right hand side of Eq. (27), and collecting the expressions together as a determinant, we have

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial / \partial u_1 & \partial / \partial u_2 & \partial / \partial u_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \tag{36}$$

The Eq. (36) gives the expression for curl in orthogonal curvilinear coordinates.

# 6 Cartesian Coordinates

In the right handed Cartesian coordinate system, the unit vectors are

$$\begin{aligned}\hat{e}_1 &= \hat{i} \\ \hat{e}_2 &= \hat{j} \\ \hat{e}_3 &= \hat{k}.\end{aligned} \tag{37}$$

Further the components are

$$\begin{aligned}u_1 &= x \\ u_2 &= y \\ u_3 &= z.\end{aligned} \tag{38}$$

Hence the position vector in this system can be represented as

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z. \tag{39}$$

## 6.1 Metric Coefficients and Scale Factors

The metric coefficients for the orthogonal curvilinear coordinate system is given by Eq. (9)

$$g_{ij} = \sum_{i=1}^{3} \left( \frac{\partial r_k}{\partial u_i} \right)^2 . \tag{40}$$

Hence substituting Eq. (39) in this, the metric coefficients $g_{11}$, $g_{22}$ and $g_{33}$ for the Cartesian coordinate system can be evaluated as

$$\begin{aligned} g_{11} &= \left(\frac{\partial x}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 = 1 \\ g_{22} &= \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial y}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 \\ g_{33} &= \left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2 = 1. \end{aligned} \tag{41}$$

We know the scale factors are given in terms of metric coefficients as

$$\begin{aligned} h_1 &= \sqrt{g_{11}} \\ h_2 &= \sqrt{g_{22}} \\ h_3 &= \sqrt{g_{33}}. \end{aligned} \tag{42}$$

Substituting Eqs. (41), in the above equations, the scale factors for the cartesian coordinate system are

$$\begin{aligned} h_1 &= 1 \\ h_2 &= 1 \\ h_3 &= 1. \end{aligned} \tag{43}$$

## 6.2 Gradient

Let $\Phi(x, y, z)$ be a single valued scalar function in cartesian coordinate system. Then using Eqs. (37), (38) and (43), the general expression for the gradient

$$\nabla\Phi = \left\{ \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3} \right\} \Phi. \tag{44}$$

becomes

$$\nabla\Phi = \hat{i}\frac{\partial\Phi}{\partial x} + \hat{j}\frac{\partial\Phi}{\partial y} + \hat{k}\frac{\partial\Phi}{\partial z}. \tag{45}$$

## 6.3 Divergence

A vector $\vec{A}$ in general orthogonal curvilinear coordinates is given as

$$\vec{A} = \hat{e}_1 A_1 + \hat{e}_2 A_2 + \hat{e}_3 A_3. \tag{46}$$

Using Eq. (37) and assuming $A_1 = A_x$, $A_2 = A_y$ and $A_3 = A_z$ the above equation becomes

$$\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z. \tag{47}$$

The expression for the divergence in a general curvilinear system is given by Eq. (20)

$$\nabla.\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial\left(h_2 h_3 A_1\right)}{\partial u_1} + \frac{\partial\left(h_3 h_1 A_2\right)}{\partial u_2} + \frac{\partial\left(h_1 h_2 A_3\right)}{\partial u_3}\right]. \tag{48}$$

Using Eqs. (38) and (43) and assuming $A_1 = A_x$, $A_2 = A_y$ and $A_3 = A_z$, the divergence of vector in cartesian coordinate system is given as

$$\nabla.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{49}$$

## 6.4 Laplacian

The general expression for the Laplacian of a scalar function $\Phi$ in general orthogonal curvilinear coordinate system is given from Eq. (24) as

$$\nabla^2\Phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right] \tag{50}$$

Using Eqs. (38) and (43), the Laplacian of the scalar function in cartesian coordinate system is given as

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}. \tag{51}$$

### 6.5 Curl

The general equation for the curl of a vector $\vec{A}$ in curvilinear coordiantes is given from Eq. (36) as

$$\nabla \times \vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} \tag{52}$$

Using Eqs. (37), (38) and (43), the curl of the vector $\vec{A}$ in cartesian coordinate system is given as

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} \tag{53}$$

## 7 Cylindrical Coordinates

In the cylindrical coordinate system (*or the right circular cylindrical coordinate system*), the unit vectors are

$$\begin{aligned} \hat{e}_1 &= \hat{e}_\rho \\ \hat{e}_2 &= \hat{e}_\phi \\ \hat{e}_3 &= \hat{e}_z. \end{aligned} \tag{54}$$

and the coordinate axes are

$$
\begin{aligned}
u_1 &= \rho \\
u_2 &= \phi \\
u_3 &= z.
\end{aligned} \tag{55}
$$

Hence a position vector in this system can be represented as

$$\vec{r} = \hat{e}_\rho(\rho\ cos\phi) + \hat{e}_\phi(\rho\ sin\phi) + \hat{e}_z z. \tag{56}$$

Hence the components of a vector in this system are

$$
\begin{aligned}
r_1 &= \rho\ cos(\phi) \\
r_2 &= \rho\ sin(\phi) \\
r_3 &= z.
\end{aligned} \tag{57}
$$

## 7.1 Metric Coefficients and Scale Factors

The metric coefficients for the orthogonal curvilinear coordinate system are given by Eq. (9) as

$$g_{ij} = \sum_{i=1}^{3} \left( \frac{\partial r_k}{\partial u_i} \right)^2. \tag{58}$$

Using Eqs. (55), (56) and (57), the metric coefficients become

$$\begin{aligned}
g_{11} &= \left\{\frac{\partial(\rho\, cos(\phi))}{\partial\rho}\right\}^2 + \left\{\frac{\partial(\rho\, sin(\phi))}{\partial\rho}\right\}^2 + \left\{\frac{\partial z}{\partial\rho}\right\}^2 \\
&= cos^2\phi + sin^2\phi \\
&= 1 \\
g_{22} &= \left\{\frac{\partial(\rho\, cos(\phi))}{\partial\phi}\right\}^2 + \left\{\frac{\partial(\rho\, sin(\phi))}{\partial\phi}\right\}^2 + \left\{\frac{\partial z}{\partial\phi}\right\}^2 \\
&= \rho^2\, cos^2\phi + \rho^2\, sin^2\phi \\
&= \rho^2 \\
g_{33} &= \left\{\frac{\partial(\rho\, cos(\phi))}{\partial z}\right\}^2 + \left\{\frac{\partial(\rho\, sin(\phi))}{\partial z}\right\}^2 + \left\{\frac{\partial z}{\partial z}\right\}^2 \\
&= 1. \qquad (59)
\end{aligned}$$

Using these relations, the scale factors are given as

$$\begin{aligned}
h_1 &= 1 \\
h_2 &= \rho \\
h_3 &= 1. \qquad (60)
\end{aligned}$$

## 7.2 Gradient

Let $\Phi(\rho, \phi, z)$ be a single valued scalar function in cylindrical coordinate system. Then using Eqs. (54), (55) and (60), the general expression for the gradient

$$\nabla\Phi = \left\{\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}\right\}\Phi. \qquad (61)$$

becomes

$$\nabla\Phi = \hat{e}_\rho\frac{\partial\Phi}{\partial\rho} + \hat{e}_\phi\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} + \hat{e}_z\frac{\partial\phi}{\partial z}. \qquad (62)$$

## 7.3 Divergence

The general expression for the divergence of a vector in orthogonal curvilinear coordinates is given as

$$\nabla.\vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial\left(h_2 h_3 A_1\right)}{\partial u_1} + \frac{\partial\left(h_3 h_1 A_2\right)}{\partial u_2} + \frac{\partial\left(h_1 h_2 A_3\right)}{\partial u_3}\right], \tag{63}$$

where the vector $\vec{A}(\rho, \phi, z)$ is defined in cylindrical polar coordinates as

$$\vec{A} = \hat{e}_\rho A_\rho + \hat{e}_\phi A_\phi + \hat{e}_z A_z. \tag{64}$$

Then from the Eqs. (55) and (60) and assuming $A_1 = A_\rho$, $A_2 = A_\phi$ and $A_3 = A_z$, the divergence of the vector in the cylindrical polar coordinate system becomes

$$\nabla.\vec{A} = \frac{1}{\rho}\left[\frac{\partial\left(\rho A_\rho\right)}{\partial \rho} + \frac{\partial\left(A_\phi\right)}{\partial \phi} + \frac{\partial\left(\rho A_z\right)}{\partial z}\right]. \tag{65}$$

## 7.4 Laplacian

From the general relation for the Laplacian in the orthogonal curvilinear coordinate system given by Eq. (24), we have

$$\nabla^2\Phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \Phi}{\partial u_3}\right)\right] \tag{66}$$

Using Eqs. (55) and (60), the expression for the Laplacian in cylindrical polar coordinates becomes

$$\nabla^2\Phi = \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Phi}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial \Phi}{\partial z}\right)\right] \tag{67}$$

### 7.5 Curl

The general equation for the curl of a vector $\vec{A}$ in curvilinear coordiantes is given from Eq. (36) as

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \tag{68}$$

Using Eqs. (54), (55) and (60) the curl of the vector $\vec{A}$ in cylindrical polar coordinate system is given as

$$\nabla \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{e}_\rho & \hat{e}_\phi & \hat{e}_z \\ \partial/\partial \rho & \partial/\partial \phi & \partial/\partial z \\ A_\rho & A_\phi & A_z \end{vmatrix} \tag{69}$$

# 8 Spherical Polar Coordinates

In the Spherical Polar Coordinate System the unit vectors are

$$\begin{aligned} \hat{e}_1 &= \hat{e}_r \\ \hat{e}_2 &= \hat{e}_\theta \\ \hat{e}_3 &= \hat{e}_\phi. \end{aligned} \tag{70}$$

and the co-ordinate axes are

$$\begin{aligned} u_1 &= r \\ u_2 &= \theta \\ u_3 &= \phi. \end{aligned} \tag{71}$$

Hence the components of the position vector $\vec{r}$ in this system are

$$
\begin{aligned}
r_1 &= r\ sin\theta\ cos\phi \\
r_2 &= r\ sin\theta\ sin\phi \\
r_3 &= r\ cos\phi. \qquad (72)
\end{aligned}
$$

## 8.1 Metric Coefficients and Scale Factors

The metric coefficients for the orthogonal curvilinear coordinate system are given by Eq. (9) as

$$
g_{ij} = \sum_{i=1}^{3} \left( \frac{\partial r_k}{\partial u_i} \right)^2 . \qquad (73)
$$

Using Eqs. (71) and (72), the metric coefficients become

$$
\begin{aligned}
g_{11} &= \left\{ \frac{\partial (r\ sin\theta\ cos\phi)}{\partial r} \right\}^2 + \left\{ \frac{\partial (r\ sin\theta\ sin\phi)}{\partial r} \right\}^2 + \left\{ \frac{\partial (r\ cos\theta)}{\partial r} \right\}^2 , \\
&= sin^2\theta\ \left[ cos^2\phi + sin^2\phi \right] + cos^2\theta, \\
&= 1, \\
g_{22} &= \left\{ \frac{\partial (r\ sin\theta\ cos\phi)}{\partial \theta} \right\}^2 + \left\{ \frac{\partial (r\ sin\theta\ sin\phi)}{\partial \theta} \right\}^2 + \left\{ \frac{\partial (r\ cos\theta)}{\partial \theta} \right\}^2 , \\
&= r^2\ sin^2\theta\ \left[ cos^2\phi + sin^2\phi \right] + r^2\ cos^2\theta, \\
&= r^2, \\
g_{33} &= \left\{ \frac{\partial (r\ sin\theta\ cos\phi)}{\partial \phi} \right\}^2 + \left\{ \frac{\partial (r\ sin\theta\ sin\phi)}{\partial \phi} \right\}^2 + \left\{ \frac{\partial (r\ cos\theta)}{\partial \phi} \right\}^2 , \\
&= sin^2\theta\ \left[ cos^2\phi + sin^2\phi \right], \\
&= r^2\ sin^2\theta. \qquad (74)
\end{aligned}
$$

Using these relations, the scale factors are given as

$$\begin{aligned} h_1 &= 1 \\ h_2 &= r \\ h_3 &= r\ sin\theta. \end{aligned} \tag{75}$$

## 8.2 Gradient

Let $\Phi(r, \theta, \phi)$ be a single valued scalar function in spherical polar coordinate system. Then using Eqs. (70), (71) and (75), the general expression for the gradient

$$\nabla\Phi = \left\{ \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3} \right\} \Phi. \tag{76}$$

becomes

$$\nabla\Phi = \hat{e}_r \left( \frac{\partial \Phi}{\partial r} \right) + \hat{e}_\theta \frac{1}{r} \left( \frac{\partial \Phi}{\partial \theta} \right) + \hat{e}_\phi \frac{1}{r\ sin(\theta)} \left( \frac{\partial \Phi}{\partial \phi} \right). \tag{77}$$

## 8.3 Divergence

The general expression for the divergence of a vector in orthogonal curvilinear coordinates is given as

$$\nabla . \vec{A} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial \left( h_2 h_3 A_1 \right)}{\partial u_1} + \frac{\partial \left( h_3 h_1 A_2 \right)}{\partial u_2} + \frac{\partial \left( h_1 h_2 A_3 \right)}{\partial u_3} \right], \tag{78}$$

where the vector $\vec{A}(r, \theta, \phi)$ is defined in spherical polar coordinates as

$$\vec{A} = \hat{e}_r A_r + \hat{e}_\theta A_\theta + \hat{e}_\phi A_\phi. \tag{79}$$

Then from the Eqs. (71) and (75) and assuming $A_1 = A_r$, $A_2 = A_\theta$ and $A_3 = A_\phi$, the divergence of the vector in the spherical polar coordinate system becomes

$$\nabla . \vec{A} = \frac{1}{r^2 \, sin\theta} \left[ \frac{\partial \left( r^2 \, sin\theta A_r \right)}{\partial r} + \frac{\partial \left( r \, sin\theta A_\theta \right)}{\partial \theta} + \frac{\partial \left( r \, A_\phi \right)}{\partial \phi} \right]. \qquad (80)$$

## 8.4 Laplacian

From the general relation for the Laplacian in the orthogonal curvilinear coordinate system given by Eq. (24), we have

$$\nabla^2 \Phi = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} \left( \frac{h_2 h_3}{h_1} \frac{\partial \Phi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left( \frac{h_1 h_3}{h_2} \frac{\partial \Phi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left( \frac{h_1 h_2}{h_3} \frac{\partial \Phi}{\partial u_3} \right) \right] \qquad (81)$$

Using Eqs. (71) and (75), the expression for the Laplacian in spherical polar coordinates becomes

$$\nabla^2 \Phi = \frac{1}{r^2 \, sin\theta} \left[ \frac{\partial}{\partial r} \left( r^2 sin\theta \, \frac{\partial \Phi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( sin\theta \, \frac{\partial \Phi}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left( \frac{1}{sin\theta} \frac{\partial \Phi}{\partial \phi} \right) \right] \qquad (82)$$

## 8.5 Curl

The general equation for the curl of a vector $\vec{A}$ in curvilinear coordiantes is given from Eq. (36) as

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial / \partial u_1 & \partial / \partial u_2 & \partial / \partial u_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \qquad (83)$$

Using Eqs. (70), (71) and (75) the curl of the vector $\vec{A}$ in spherical polar coordinate system is given as

$$\nabla \times \vec{A} = \frac{1}{r^2\ sin\theta} \begin{vmatrix} \hat{e}_r & r\ \hat{e}_\theta & r sin\theta\ \hat{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & A_\theta & A_\phi \end{vmatrix} \tag{84}$$

# 9 Summary

The equivalent expressions for the various quantities in the different co-ordinate systems can be summarised in a tabular form as given.

| **Quantities** | **General Curvilinear Coordinates** | **Cartesian Coordinates** |
|---|---|---|
| Coordinates | $u_1,u_2,u_3$ | $x,y,z$ |
| Scale Factor | $h_1,h_2,h_3$ | $1,1,1$ |
| Scalar | $\Phi(u_1,u_2,u_3)$ | $\Phi(x,y,z)$ |
| Vector | $\vec{A} \equiv \vec{A}(u_1,u_2,u_3)$ | $\vec{A} \equiv \vec{A}(x,y,z)$ |
| Gradient | $\nabla\Phi = \frac{\hat{e}_1}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial\Phi}{\partial u_3}$ | $\nabla\Phi = \hat{i}\frac{\partial\Phi}{\partial x} + \hat{j}\frac{\partial\Phi}{\partial y} + \hat{k}\frac{\partial\Phi}{\partial z}$ |
| Divergence | $\nabla.\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial\,(h_2h_3A_1)}{\partial u_1} + \frac{\partial\,(h_3h_1A_2)}{\partial u_2} + \frac{\partial\,(h_1h_2A_3)}{\partial u_3}\right]$ | $\nabla.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ |
| Laplacian | $\nabla^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]$ | $\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$ |
| Curl | $\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$ | $\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$ |

| **Quantities** | **Cylindrical Polar Coordinates** | **Spherical Polar Coordinates** |
|---|---|---|
| Coordinates | $\rho$,$\phi$,$z$ | $r$,$\theta$,$\phi$ |
| Scale Factor | 1,$\rho$,1 | $1, r, r; \sin\theta$ |
| Scalar | $\Phi(\rho, \phi, z)$ | $\Phi(r, \theta, \phi)$ |
| Vector | $\vec{A} \equiv \vec{A}(\rho, \phi, z)$ | $\vec{A} \equiv \vec{A}(r, \theta, \phi)$ |
| Gradient | $\nabla\Phi = \hat{e}_\rho \frac{\partial\Phi}{\partial\rho} + \hat{e}_\phi \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi} + \hat{e}_z \frac{\partial\phi}{\partial z}$ | $\nabla\Phi = \hat{e}_r \left(\frac{\partial\Phi}{\partial r}\right) + \hat{e}_\theta \frac{1}{r}\left(\frac{\partial\Phi}{\partial\theta}\right) + \hat{e}_\phi \frac{1}{r\ sin(\theta)}\left(\frac{\partial\Phi}{\partial\phi}\right)$ |
| Divergence | $\nabla.\vec{A} = \frac{1}{\rho}\left[\frac{\partial\,(\rho A_\rho)}{\partial\rho} + \frac{\partial\,(A_\phi)}{\partial\phi} + \frac{\partial\,(\rho A_z)}{\partial z}\right]$ | $\nabla.\vec{A} = \frac{1}{r^2\ sin\theta}\left[\frac{\partial\,(r^2\ sin\theta A_r)}{\partial r} + \frac{\partial\,(r\ sin\theta A_\theta)}{\partial\theta} + \frac{\partial\,(r\ A_\phi)}{\partial\phi}\right]$ |
| Laplacian | $\nabla^2\Phi = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial\Phi}{\partial z}\right)\right]$ | $\nabla^2\Phi = \frac{1}{r^2\ sin\theta}\left[\frac{\partial}{\partial r}\left(r^2 sin\theta\ \frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(sin\theta\ \frac{\partial\Phi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{sin\theta}\ \frac{\partial\Phi}{\partial\phi}\right)\right]$ |
| Curl | $\nabla\times\vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{e}_\rho & \hat{e}_\phi & \hat{e}_z \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ A_\rho & A_\phi & A_z \end{vmatrix}$ | $\nabla\times\vec{A} = \frac{1}{r^2\ sin\theta}\begin{vmatrix} \hat{e}_r & r\ \hat{e}_\theta & r sin\theta\ \hat{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & A_\theta & A_\phi \end{vmatrix}$ |